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Neutralized Competition

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Abstract

This paper proposes a tractable competing mechanism game where each seller simultaneously posts a trading contract that specifies a menu of dominant strategy incentive compatible (DIC) direct mechanisms conditional on an array of messages sent by buyers, and each seller subsequently chooses a DIC direct mechanism from his menu. The complete set of a seller's profits that are supportable in a (symmetric) equilibrium is the interval between the minmax value of his profit with respect to DIC direct mechanisms and his profit in the joint profit maximization. The set of a seller's equilibrium profits is robust to the possibility of a seller's deviation to any arbitrary mechanism in the standard environment with linear utilities and independent private type. Further, with no limited liability or with no capacity constraints, the set of a seller's equilibrium profits coincides with the set of his feasible (i.e., individually rational and incentive compatible) profits. Given a number of buyers, the number of sellers can be endogenized and is equal to the largest number at which a seller's profit in the joint profit maximization is non-negative: As the number of buyers increases, competition is neutralized because only the monopoly terms of trade prevails in the market, whereas the range of a seller's equilibrium profits shrinks to his reservation profit. (JEL C72, D47, D82)

1 Introduction

The neoclassical economic theory provides a well-known wisdom that competition among firms drives down the equilibrium price to marginal cost; In the

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short-run, firms produce as long as the price is at least as high as the average variable cost. The number of firms is endogenously determined in the long run - at the point where the price is equal to the average cost - so that a firm's equilibrium profit is simply zero. The standard game-theoretic approach, Bertrand or Cournot, confirms the wisdom. Unfortunately, the neoclassical economic theory completely ignores the effect of contracting between firms and buyers on equilibrium outcomes and the standard game-theoretical analysis also restricts a firm's strategy space into either prices or quantities.

In practice, as buyers search for a better deal, they are informed about terms of trade or prices offered by sellers in the market. Not only do they have private information on their payoff type, this implies that they also have market information. It is important for a seller to gather market information from buyers to determine his terms of trade.

It would have been a daunting task to gather market information from buyers at the time in which the neoclassical economic theory or standard game-theoretic competition analysis was established. However, it is quite easy to gather market information today due to the rapid advance in technology. This is an especially prominent feature of a seller's web design in on-line markets. On-line sellers can keep track of buyers' search history based on html cookies and most of them are based on binary messages: Whether or not a buyer revisits a seller's web site, whether or not a buyer clicks a certain part of a seller's web site, etc. This type of information can reveal what buyers know about competing sellers' terms of trade. For example, the more buyers revisit a seller's web site, the more likely it reflects their intensified search from finding a lower price somewhere else (Peters 2015). The number of web sites that a buyer has visited can also reflect how well a buyer is informed about the products in the market (Board and Lu, 2015). This requires a new competition theory that shows how firms incorporate buyers' market information in determining their terms of trade and how it affects equilibrium outcomes.

Despite the prevalent use of buyers' search behavior or their market information in practice, it is extremely difficult to develop a tractable competition theory that reflects it. In particular, the standard revelation principle that has been adopted in mechanism design no longer holds in the market where sellers post trading contracts (or mechanisms) to compete for trading opportunities. The message space must be enlarged so that it allows buyers to reveal not only their payoff type but also their market information. Sellers can then maintain many collusive outcomes if they commit themselves to change their terms of trade in order to punish a deviating seller when a competing seller's deviation becomes evident from buyers' messages.

The big idea - that mechanisms can involve commitments to punish deviation

- has been explored in the literature on competing mechanisms. Punishment can take place upon a competing seller's deviation revealed by buyers' messages, or it is prescribed in a mechanism, directly conditional on a deviator's publicly observable contract. The message space needed to reflect the buyer's market information or the language needed to make punishment directly conditional on the deviator's contract is however complicated. For examples, an infinite sequence of real numbers is involved to describe the deviator's mechanism (Epstein and Peters, 1999), a message should be at least as complex as the entire mapping of a direct mechanism (Yamashita, 2010, Peters and Troncoso-Valverde, 2013), or the deviator's contract should be directly defined in a non-deviator's mechanism or contract by using the Godel language (Peters and Szentes (2012), Szentes (2015)). Further, the set of robust equilibrium payoffs is defined with the minmax value with respect to such complex mechanisms or contracts.¹

The literature on competing mechanisms mentioned above assumes that a profile of messages sent by buyers determines a seller's action (i.e., terms of trade or his allocation) given his contract or mechanism. This means that a seller completely delegates his action choice to buyers given his contract or mechanism. Szentes (2010) shows that it is controversial. If a deviating seller's contract is flexible enough to let a message profile determine a subset of actions (i.e., a menu of actions) from which the deviating seller can choose his action, it leads to the full characterization of the set of equilibrium allocations in the complete information case, with the threshold of a seller's (ex-ante) expected payoff equal to his minmax value with respect to the actions. It seems more general to allow a seller to design a contract that specifies a menu of actions he can take conditional on a message profile sent by buyers. However, it is not yet known how to apply the idea of Szentes' approach beyond the complete information case.

This paper extends the idea of Szentes' approach to the incomplete information case where buyers' payoff types are their private information. One might think that it would be enough for a seller to offer a contract that specifies a subset (i.e., a menu) of Bayesian incentive compatible (BIC) direct mechanisms contingent on a message profile sent by buyers. It seems natural, but it does not work well because the Bayesian incentive compatibility of a seller's direct mechanism is endogenous because, for example, it depends on the number of buyers who select the seller, which depends on what contracts or mechanisms

¹Xiong (2013) proposes a mechanism with a unit interval as a buyer's message space. However, it is hard to apply his approach because his equilibrium analysis is based on a mixed-strategy continuation equilibrium for agents' communication. It restates Yamashita's results based on a mixed-strategy continuation equilibrium without characterizing it, as Yamashita (2010) does not characterize a pure-strategy continuation equilibrium.

are offered by other sellers as well. We propose a competing mechanism game G where each seller posts a contract that specifies a menu of dominant strategy incentive compatible (DIC) direct mechanisms contingent on a message profile sent by buyers. Since dominant strategy incentive compatibility can be ensured without reference to other sellers' contracts or mechanisms, a DIC direct mechanism can be freely offered or chosen by a seller. Once a menu of DIC direct mechanisms is determined, a seller chooses a DIC mechanism that he wants from the menu and then it implements his action contingent on the type messages sent by participating buyers.

The set of a seller's (symmetric) equilibrium payoffs in this competing mechanism game, denoted by Φ_J^e , is shown to be a connected interval between the lower bound and upper bound.² The lower bound is the minmax value of a seller's expected profit with respect to DIC direct mechanisms and the upper bound is the expected profit achieved by the joint profit maximization. In addition, any profit in Φ_J^e can be supported in an equilibrium where a message set for a buyer in a seller's contract has only two messages (one and zero). We call a contract with binary messages a deviation-reporting contract.³ This is a practically important feature because binary messages can be easily adopted in practice (e.g., on-line markets).

Further, we show that the BIC-DIC equivalence established by Gershkov, et al. (2013) for a single principal is extended to multiple principals (sellers) in the standard environment with linear utilities and independent, one-dimensional, private payoff types. Given this BIC-DIC equivalence, we show that Φ_J^e is the set of a seller's robust equilibrium profits in the sense that a seller cannot gain in any continuation equilibrium upon his deviation to any arbitrary mechanism.

One might ask when the set of a seller's robust equilibrium payoffs Φ_J^e is the same as the set of a seller's feasible (i.e., individually rational and incentive compatible) payoffs, denoted by Φ_J^* , with a seller's reservation profit as the lower bound of his payoff. We show that if a seller has no limited liability, then $\Phi_J^e = \Phi_J^*$. However, if a seller has limited liability so that he cannot transfer a positive amount of money to a buyer, the equivalence between Φ_J^e and Φ_J^* depends on whether a seller has capacity constraints or not.

We present two applications where sellers with constant marginal cost have limited liability and each buyer has a unit demand with private valuation: one with no capacity constraints and the other with a capacity constraint. If sellers

²The subscript J in Φ_J^e denotes the number of sellers in the market.

³If a majority of buyers report 1 in this deviation-reporting contract, it is interpreted as a competing seller's deviation, and the contract assigns a single DIC mechanism that punishes the deviator. Otherwise, it continues to assign a single DIC mechanism that is supposed to be offered in an equilibrium. This result is formally shown later.

have no capacity constraint, they can lower their price down to their marginal cost upon a competing seller's deviation reported in their deviation-reporting contracts. This implies that any seller must incur loss in order to attract buyers upon deviation to any arbitrary mechanism. Therefore, the lower bound of Φ_J^e can be as low as a seller's reservation profit and hence $\Phi_J^e = \Phi_J^*$. We show that any profit level in Φ_J^* can be implemented by the deviation-reporting contract that only assigns a single price depending on buyers' binary messages: If the hazard rate of the buyer's valuation is non-decreasing, the upper bound of Φ_J^* is reached at the monopoly price, i.e., the buyer's valuation that makes her virtual valuation equal to zero. This analysis is based on the fixed number of buyers and sellers.

However, sellers may be free to enter the market by incurring a fixed cost. As long as the number of sellers does not exceed the level where even the joint profit maximization yields a negative net profit given the number of buyers, a seller's profit is no less than his reservation profit in any equilibrium. Therefore, the number of sellers is determined at the point where one additional seller in the market yields a negative net profit even in joint profit maximization. Given a finite number of buyers, the equilibrium number of sellers may still generate a non-degenerate range of equilibrium profits and the corresponding range of prices. However, as the number of buyers goes to infinity, the equilibrium ratio of buyers to sellers yields the monopoly price as the unique equilibrium price, although their equilibrium profit is zero.

What if each seller can produce at most a single unit? This is a canonical environment for competing auctions (Burguet and Sákovics (1999), Han (2015), McAfee (1993), Peters and Severinov (1997), Peters (1997), Virag (2010)). If the hazard rate of a buyer's valuation is non-decreasing, we can show that the upper bound of Φ_J^e can be reached when every seller offers the monopoly auction, i.e., one with reserve price equal to the buyer's valuation at which her virtual valuation is zero. However, the lower bound of Φ_J^e is a lot more complex because of limited liability and capacity constraints. Even if non-deviating sellers give away their product for free, capacity constraints imply that each buyer who selects a non-deviating seller faces a positive probability that she does not get the object. Therefore, if a deviating seller offers a price slightly higher than zero, buyers would choose him with less likelihood than they would choose a non-deviator. This implies that the lower bound of Φ_J^e would be generally higher than a seller's reservation profit, but it is still very difficult to identify the lower bound of Φ_J^e .

The lower bound of Φ_J^e is the minmax value of a seller's expected profit with respect to all DIC direct mechanisms. However, it is difficult to derive the minmax value with respect to all DIC direct mechanisms with limited liability and

capacity constraints. We instead specify a lower bound of a seller's equilibrium profit as his minmax value with respect to all reserve prices of (second-price) auctions. This implies that non-deviating sellers punish a deviating seller with auctions with reserve price and a deviating seller himself also restricts his deviation to auctions with reserve price. This creates one difficulty: Can a deviating seller do better with a mechanism other than an auction, given non-deviating sellers' auctions? To answer this question, we can use the results in Peters (1997) and Han (2015). Peters (1997) shows that given any distribution of auctions in a large market, a single seller cannot do better with any direct mechanism other than an auction with reserve price equal to his cost. We can then apply the result in Han (2015) to show that a single seller cannot do better with any arbitrary mechanism other than any direct mechanism given any distribution of auctions. Therefore, it is acceptable to restrict a seller's deviation only to auctions with reserve price at least in a large market.⁴

Note that the lower bound of a seller's equilibrium profit is expressed as his minmax value with respect to reserve prices of auctions.⁵ Further, the upper bound of a seller's equilibrium profit is achieved by auctions with reserve price when the hazard rate of the buyer's valuation is non-decreasing. Therefore, it is easy to show that any profit in between can be achieved by some level of the reserve price. Such a profit can be sustained in a robust equilibrium where a seller's deviation-reporting contract specifies his reserve price conditional on buyers' binary messages on whether a competing seller deviates. In this large market for auctions, we also endogenize the ratio of buyers to sellers, assuming sellers can enter the market by spending a fixed cost. As long as the ratio is high enough to ensure a non-negative net profit associated with the monopoly auction (i.e., profit in the joint profit maximization), sellers keep entering the market because any equilibrium net profit is non-negative in that case. The equilibrium ratio of buyers to sellers is determined at the point where the equilibrium net profit is zero and the unique equilibrium reserve price is the same as one for the monopoly auction.

⁴This result is not independent of the probability distribution F of the buyer's valuation x . It is not clear whether an auction is a seller's best response to other sellers' auctions even when a seller can choose any arbitrary selling mechanism. Virag (2007) shows that in a market with two sellers and two buyers, that's the case when the probability density function of the buyer's valuation f , and $x + \frac{F}{f}$ are both increasing.

⁵In our numerical exercise for a large market, this minmax value is equal to a seller's profit when his reserve price is zero given that all other sellers' reserve prices are also zero regardless of the ratio of buyers to sellers. This happens to be an equilibrium when sellers are restricted to choose only reserve prices for their auctions so that the minmax and Nash-reversion punishment are identical. Our numerical exercise shows that it is not true in a finite market.

Our results suggest that a wide range of equilibrium profits can be supported in the short run with a fixed number of buyers and sellers with or without capacity constraints as long as sellers find a way to raise revenue that can cover their average variable cost. However, the ratio of buyers and sellers is endogenously determined in the long run and the monopoly terms of trade uniquely prevail in a large market with or without capacity constraints, but the seller's equilibrium profit is equal to his reservation profit. This clearly shows how sophisticated trading can completely neutralize the beneficial effect of competition. This type of competition was not foreseen when the neoclassical economic theory or standard game-theoretic competition model was established because of the technological constraints of the time. Therefore, we need to rethink about the nature of competition and the role of the market in the context of today's technology.

2 Preliminaries

J principals ($J \geq 2$, e.g., sellers) compete in the market with N agents (e.g., buyers) with $N \geq 3$. Each agent's type is independently drawn from a probability distribution F with support $X = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$.⁶ Each principal makes an allocation decision for agents who select him. We assume that each principal's allocation decision is to choose (i) one of his action alternatives from the finite set, $\mathcal{K} = \{1, \dots, K\}$, and (ii) monetary transfers to agents who select him.

An agent's payoff depends on the action taken by the principal she selects and monetary transfer. The payoff for an agent of type x associated with choosing a principal who takes action alternative k is

$$b^k x + g^k + t, \tag{1}$$

where $b^k \geq 0$ and $g^k \in \mathbb{R}$ for all $k \in \mathcal{K}$ and $t \leq 0$ is a monetary transfer to the agent.⁷ For example, k is the identity of the winning bidder in an auction environment. In this case, bidder n has the following preference parameter values: $b^k = 1$ if $k = n$, $b^k = 0$ otherwise; $g^k = 0$ for all k . If an agent does not choose any principal, she receives her reservation payoff, which is equal to zero.

⁶We use masculine pronouns for principals and feminine pronouns for agents.

⁷We consider the cases where an agent (buyer) pays a positive amount of money to a principal (seller). Because t is a monetary transfer to the agent, it means that t is assumed to be non-positive. The assumption of $t \leq 0$ essentially implies that a principal (seller) cannot make a positive amount of monetary transfer to the agent. This is what we usually observe in practice. We can assume $t \geq 0$ for the cases where principals are buyers and agents are sellers.

A principal's payoff depends on his choice of action alternative k and monetary transfer ℓ ;

$$a^k w + y^k + \ell, \quad (2)$$

where $a^k, y^k \in \mathbb{R}$ for all $k \in \mathcal{K}$, $w \in \mathbb{R}$. We assume that $\max_k [a^k w + y^k] \geq 0$.

For an auction environment, we can set $\mathcal{K} = \{1, \dots, N, N+1\}$ so that, given N potential bidders, alternative k implies that the winning bidder is bidder k if $k \leq N$, but $k = N+1$ means that the seller (principal) retains the object. The seller has then the following preference parameter values: $a^k = 0$ for $k \leq N$ and $a^k = 1$ for $k = N+1$; $y^k = 0$ for all k . w represents the value of the object to the seller. A principal's reservation payoff is equal to zero. We impose the *budget balance* condition for each principal so that the sum of ℓ and monetary transfers t to agents who choose him is equal to zero. This implies that $\ell \geq 0$. Non-positive monetary transfer to an agent and non-negative monetary transfer to a principal reflect limited liability on both agents and principals.

3 Feasible Allocations

An allocation is characterized by (a) an array of direct mechanisms offered by principals and (b) the agent's selection behavior that decides how to select a principal among J principals given an array of direct mechanisms.

If an agent does not select a principal, he treats her type as x° . Let $\bar{X} = X \cup \{x^\circ\}$. For any given principal, $\mathbf{x} \in \bar{X}^N$ can conveniently characterize the type profile of the agents who select him. Let a principal's direct mechanism μ be denoted by $\{q^1, \dots, q^K, t\}$, where $q^k : \bar{X}^N \rightarrow [0, 1]$ determines the probability of alternative k as a function of agents' type messages and $t : \bar{X}^N \rightarrow \mathbb{R}_-$ determines an amount of monetary transfer to an agent as a function of agents' type messages. We assume that mechanisms are *anonymous* with respect to agents so that they cannot distinguish among different agents except on a basis of messages sent by agents to the mechanisms. Principals are also not distinguished except on the basis of their mechanisms (and contracts). We are interested in a (*symmetric*) *allocation* where (i) agents choose principals with equal probability as long as their mechanisms are the same and (ii) every principal's direct mechanism is identical.

We first construct an agent's selection behavior given an array of direct mechanisms when one principal offers μ' and all the other principals offer μ . Let $\pi(\mu', \mu)(x) \in [0, 1]$ denote the probability with which each agent of type x selects the principal who offers μ' when all the other principals offer μ . Define

$z(\pi(\mu', \mu))(x)$ as follows

$$z(\pi(\mu', \mu))(x) := 1 - \int_x^{\bar{x}} \pi(\mu', \mu)(s) dF. \quad (3)$$

The term $z(\pi(\mu', \mu))(x)$ is the probability that an agent either has her type below x as a participant of μ' or selects any other principal whose mechanism is μ .

An array of DIC direct mechanisms (μ', μ) chosen by principals defines a subgame that agents play.⁸ A (truthful symmetric) *continuation equilibrium* at a subgame (μ', μ) can be characterized by (a) every agent's selection strategy $\pi(\mu', \mu)$ and (b) Bayesian incentive compatibility of direct mechanisms.⁹ Let $\Pi(\mu', \mu)$ be the set of all possible continuation equilibria (i.e., the set of all optimal selection strategies that agents choose in all continuation equilibria at a subgame (μ', μ)). Given $\mu' = \{q^1, \dots, q^K, t\}$, we can derive the reduced-form direct mechanism $\{\dot{Q}^1, \dots, \dot{Q}^K, T\}$ such that, for all $x \in X$,

$$\dot{Q}^k(x) := \int_{\underline{x}}^{\bar{x}} \dots \int_{\underline{x}}^{\bar{x}} \dot{q}^k(x, s_2, \dots, s_I) dz(\pi(\mu', \mu))(s_2) \dots dz(\pi(\mu', \mu))(s_I), \quad (4)$$

$$\dot{T}(x) := \int_{\underline{x}}^{\bar{x}} \dots \int_{\underline{x}}^{\bar{x}} \dot{t}(x, s_2, \dots, s_I) dz(\pi(\mu', \mu))(s_2) \dots dz(\pi(\mu', \mu))(s_I). \quad (5)$$

The interim expected payoff for the agent of type x associated with reporting the true type upon selecting the principal whose direct mechanism is μ' is

$$U_J(\mu', \mu, \pi, x) := \sum_{k \in \mathcal{K}} (b^k x + g^k) \dot{Q}^k(x) + \dot{T}(x). \quad (6)$$

μ' is Bayesian incentive compatible (BIC) given $\pi(\mu', \mu)$ if for all $x, x' \in X$

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) \dot{Q}^k(x) + \dot{T}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) \dot{Q}^k(x') + \dot{T}(x') \quad (7)$$

The ex-ante expected payoff for the principal whose direct mechanism is μ' is¹⁰

$$\Phi_J(\mu', \mu, \pi) := \sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \dot{Q}^k(s_1) dz(\pi(\mu', \mu))(s_1) - N \int_{\underline{x}}^{\bar{x}} \dot{T}(s_1) dz(\pi(\mu', \mu))(s_1). \quad (8)$$

We define a (symmetric) feasible allocation as follows.

⁸ (μ', μ) means that one principal offers μ' and all the other principals offer μ .

⁹In a (symmetric) continuation equilibrium at a subgame (μ', μ) , an agent of type x selects a principal whose mechanism is μ with probability $\frac{1 - \pi(\mu', \mu)(x)}{J-1}$ if she selects a principal with μ .

¹⁰The subscript in $\Phi_J(\mu', \mu, \pi)$ and $U_J(\mu', \mu, \pi, x)$ denotes the number of principals in the market.

Definition 1 (μ, π) is a (symmetric) feasible allocation if (i) μ is BIC given $\pi(\mu, \mu)$, (ii) an agent chooses each principal according to $\pi(\mu, \mu)(x) = \frac{1}{J}$ whenever $\pi(\mu, \mu)(x) > 0$. (iii) for all $x \in X$, $\pi(\mu, \mu)(x) > 0$ if $U_J(\mu, \mu, \pi, x) \geq 0$, and (iv) $\Phi_J(\mu, \mu, \pi) \geq 0$.

In a feasible allocation, conditions (i) and (ii) imply that agents must play a continuation equilibrium in the subgame which all principals' mechanisms are identical. Conditions (iii) and (iv) imply that a feasible allocation must be *individually rational* for both agents and principals. Let Z be the set of all feasible allocations.

We are first interested in the set of a principal's ex-ante expected payoffs associated with all possible feasible allocations in Z . It cannot be lower than a principal's reservation payoff, which is zero. Therefore, the minimum of a principal's ex-ante expected payoff is zero. Given symmetry in allocation, the maximum of a principal's ex-ante expected payoff can be derived by solving the joint payoff maximization problem.

There may be an agent who decides not to choose any principal given a direct mechanism μ chosen by all principals. If that's the case, we assume that an agent first selects one of the principals with equal probability $1/J$ and sends the type message x° . Therefore, in the joint payoff maximization problem, we fix $\pi(\mu, \mu)(x) = \frac{1}{J}$ for all $x \in X$. Then,

$$z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x \in X. \quad (9)$$

Given a direct mechanism $\mu = \{q^1, \dots, q^K, t\}$, let $\{Q^1, \dots, Q^K, T\}$ be the reduced-form direct mechanism that is derived according to (4) and (5) with (9). Then, the joint payoff maximization problem is

$$\max_{\mu} \Phi_J(\mu, \mu, \pi) \quad (10)$$

subject to

$$\begin{aligned} \text{(IC)} \quad & \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x) + T(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x') + T(x') \quad \forall x, x' \in X, \\ \text{(IR)} \quad & \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x) + T(x) \geq 0 \quad \forall x \in X, \end{aligned}$$

and $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in \bar{X}^N$. We assume that a solution, denoted by $\bar{\mu} = \{\bar{q}^1, \dots, \bar{q}^K, \bar{t}\}$, to the joint payoff maximization problem exists and let $\bar{\phi}_J := \Phi_J(\bar{\mu}, \bar{\mu}, \pi)$. Let Φ_J^* be the set of the principal's feasible ex-ante expected payoffs:

$$\Phi_J^* := \{\phi \in \mathbb{R} : \phi = \Phi_J(\mu, \mu, \pi) \quad \forall (\mu, \pi) \in Z\}.$$

Theorem 1 $\Phi_J^* = [0, \bar{\phi}_J]$.

Proof. It is clear that a principal's ex-ante expected payoff associated with a feasible allocation cannot be less than zero and greater than $\bar{\phi}_J$. We only need to show that Φ_J^* is the connected interval between the two. Note that zero payoff can be achieved by

$$\mu_o := \{q_o^1, \dots, q_o^K, t_o\} \text{ with } q_o^k(\mathbf{x}) = t_o(\mathbf{x}) = 0 \ \forall (k, \mathbf{x}) \in \mathcal{K} \times \bar{X}^N. \quad (11)$$

For any $\phi \in [0, \bar{\phi}_J]$, there exists a scalar $\alpha \in [0, 1]$ such that $\phi = \alpha \bar{\phi}_J$. Given α , construct a direct mechanism

$$\mu = (1 - \alpha)\mu_o + \alpha\bar{\mu}. \quad (12)$$

Then, $(\mu, \pi) \in Z$ with π based on (9). This is because any convex combination between two BIC direct mechanisms given the same type distribution is also BIC. Accordingly, a principal's ex-ante expected payoff is

$$\Phi_J(\mu, \mu, \pi) = \alpha \Phi_J(\bar{\mu}, \bar{\mu}, \pi) = \alpha \bar{\phi}_J.$$

Therefore, the set of the principal's feasible ex-ante expected payoffs is the closed connected interval between 0 and $\bar{\phi}_J$. ■

The Bayesian incentive compatibility of a principal's direct mechanism is based on the agent's interim expected payoff, which depends on the probability that agents select the principal. Because this selection probability depends on the other principals' mechanisms as well, Bayesian incentive compatibility generally depends on the other principals' mechanisms, which makes working with BIC difficult in a market with multiple principals. On the other hand, a dominant strategy incentive compatible (DIC) direct mechanism can be easily used because the dominant strategy incentive compatibility does not depend on agents' selection strategies. A direct mechanism $\mu = \{q^1, \dots, q^K, t\}$ is DIC if for all $x, x' \in X$ and all $\mathbf{x}_{-1} = (x_2, \dots, x_I) \in \bar{X}^{N-1}$

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) q^k(x, \mathbf{x}_{-1}) + t(x, \mathbf{x}_{-1}) \geq \sum_{k \in \mathcal{K}} (b^k x' + g^k) q^k(x', \mathbf{x}_{-1}) + t(x', \mathbf{x}_{-1}).$$

As shown above, dominant strategy incentive compatibility is based on the agent's ex-post payoff after agents select the principal and hence a DIC direct mechanism can be defined without reference to other principals' mechanisms. Let Ω_D be the set of all DIC direct mechanisms. By using the BIC-DIC equivalence in Gershkov, et al. (2013), we can show that any payoff in Φ_J^* can be supported by a DIC allocation.

Corollary 1 *For any $(\tilde{\mu}, \tilde{\pi}) \in Z$, there exists a DIC allocation $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$ such that*

$$\begin{aligned} U_J(\mu, \mu, \pi, x) &= U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x) \text{ for all } x, \\ \Phi_J(\mu, \mu, \pi) &= \Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}). \end{aligned}$$

Therefore, any ϕ in Φ_J^ can be supported by a DIC allocation, i.e., $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$.*

Proof. See Appendix A. ■

Corollary 1 shows that we can focus on a DIC allocation for any feasible ex-ante expected payoff in Φ_J^* . It is also worth noting that Corollary 1 holds whether or not agents and principals have limited liability. When they have limited liability, as specified in Section 2, we need to show that $\tilde{\mu}$ also satisfies limited liability given μ that satisfies limited liability.

The next section proposes a competing mechanism game where a principal assigns a set of DIC direct mechanisms (i.e., a menu of DIC direct mechanisms) conditional on agents' market information. Each principal then chooses a DIC direct mechanism that he likes from the menu. Finally, agents choose a principal after observing DIC direct mechanisms chosen by principals.

4 Competing Mechanisms

Let $\mathcal{P}_D$ be the set of all (closed) subsets of Ω_D . One can think of a typical element in $\mathcal{P}_D$ as a menu of DIC direct mechanisms. Following the terminology in Szentes (2009), we formulate a principal's (anonymous) *contract*, which is a mapping $g : H^N \rightarrow \mathcal{P}_D$, where H is a set of messages available for each agent. Let G be the set of all possible contracts available for each principal. A competing mechanism game G unfolds in five stages as follows:

- S1. Principals simultaneously post their contracts.
- S2. Given an array of contracts, agents simultaneously send their messages in H to all principals.
- S3. Each principal simultaneously chooses a DIC direct mechanism from the menu that is determined by an array of messages sent by agents given his contract.
- S4. After observing an array of DIC direct mechanisms, each agent selects only one principal, if any, and sends her true type to the principal she selects.

S5. Allocation decisions are determined by DIC direct mechanisms according to type messages and payoffs are realized.

Our solution concept for an equilibrium of a competing mechanism game G is a symmetric pure-strategy Perfect Bayesian equilibrium where (i) principals use the same pure strategies and (ii) agents play the best (symmetric) continuation equilibrium for a deviating principal upon his deviation. From now on, we simply call it an equilibrium unless specified. The first requirement implies that principals use the same equilibrium strategy on the equilibrium path and that non-deviating principals use the same strategy to punish a deviating principal. The second requirement means that there is no continuation equilibrium where a principal gains upon his deviation. We are interested in the set of a principal's ex-ante expected payoffs that can be supported in an equilibrium of a competing mechanism game G .

We first construct a selection behavior in stage 4 when one principal's DIC direct mechanism is μ' and the other principals all use a DIC direct mechanism μ . We assume that $\Phi_J(\mu', \mu, \pi)$, Ω_D , and $\Pi(\mu', \mu)$ satisfy the properties, which ensures that $\underline{\phi}_J$ is well defined as follows:

$$\underline{\phi}_J := \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \quad (13)$$

Theorem 2 $\Phi_J^e := [\underline{\phi}_J, \bar{\phi}_J]$ is the complete set of a principal's ex-ante expected payoffs that can be supported in an equilibrium of a competing mechanism game G .

Proof. We first prove that a DIC allocation, (μ^*, π^*) with $\mu^* \in \Omega_D$, can be supported in an equilibrium of a competing mechanism game G if and only if

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \underline{\phi}_J \quad (14)$$

We prove the “only if” part by contradiction. Suppose that (μ^*, π^*) is supported in an equilibrium of a competing mechanism game G but

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \underline{\phi}_J = \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right] \quad (15)$$

Suppose that a principal deviates to a contract g such that $g(h) = \Omega_D$ for all $h \in H^N$. Given a DIC direct mechanism μ that each non-deviating principal chooses, the deviator can then choose μ' from Ω_D such that

$$\mu' \in \arg \max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right). \quad (16)$$

Combining (15) and (16) yields

$$\begin{aligned}\Phi_J(\mu^*, \mu^*, \pi^*) &< \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right] \\ &\leq \max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right).\end{aligned}$$

This implies that there exists a continuation equilibrium where a principal gains upon deviation. This contradicts that (μ^*, π^*) is supported in an equilibrium of a competing mechanism game G .

Now we prove that if (μ^*, π^*) satisfies (14), then it can be supported in a pure-strategy equilibrium of a competing mechanism game G . Divide the message space H of g to two non-empty disjoint subsets, B and B^c such that $B \cup B^c = H$. Let h_i denote agent i 's message in H . All principals post a contract g^* such that

$$g^*(h_1, \dots, h_n) := \begin{cases} \mu_p \in \Omega_D & \text{if } |\{i : h_i \in B\}| > N/2, \\ \mu^* \in \Omega_D & \text{otherwise,} \end{cases} \quad (17)$$

where μ_p satisfies

$$\mu_p \in \arg \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \quad (18)$$

If both principals post g^* , then all agents send messages in B^c and g^* assigns μ^* . Agents select principals according to $\pi^*(\mu^*, \mu^*)$ and send their true types to a principal that they select. If one principal deviates to g , then all agents send messages in B to the non-deviating principal. The non-deviator's contract g^* then assigns μ_p . Because (μ^*, π^*) satisfies (14), the deviating principal cannot gain by choosing any DIC direct mechanism, given each non-deviator's DIC direct mechanism μ_p , satisfying (18).

What we proved above is that any DIC allocation that generates a principal's ex-ante expected payoff no less than $\underline{\phi}_J$ can be supported in a pure-strategy equilibrium. Clearly, $\bar{\phi}_J$ is the maximum. We need to prove that any payoff between $\underline{\phi}_J$ and $\bar{\phi}_J$ is supportable in an equilibrium of a competing mechanism game G to complete the proof. Note that $\Phi_J^e = [\underline{\phi}_J, \bar{\phi}_J] \subset \Phi_J^*$ because $\underline{\phi}_J \geq 0$. According to Corollary 1, any ϕ in Φ_J^e can be induced by some DIC allocation, $(\mu^*, \pi^*) \in Z$ with $\mu^* \in \Omega_D$. Therefore, any ϕ in Φ_J^e can be supported in an equilibrium where every principal posts a contract g^* that assigns the corresponding μ^* when no principal deviates. ■

If a principal contract specifies a single DIC direct mechanism, instead of a menu of DIC direct mechanisms, conditional on the agent's messages, it reduces

the lower bound of a principal's equilibrium ex-ante expected payoff down to the maxmin value instead of the minmax. This is because, given the deviator's DIC direct mechanism, non-deviators choose their DIC direct mechanisms to minimize the deviator's payoff.

In a competing mechanism game G described above, a principal's contract specifies a menu of DIC direct mechanisms conditional on the agents' messages, from which he chooses his own DIC direct mechanism. This makes it possible for a deviating principal to best respond to a profile of non-deviators' DIC direct mechanisms with his DIC direct mechanism and hence the lower bound of a principal's equilibrium ex-ante expected payoff in a competing mechanism game G rises to the minmax value in terms of DIC direct mechanisms (based on the best continuation equilibrium for him).

This is the same intuition that Szentes (2010) uses to show why the lower bound of a principal's equilibrium payoff can be equal to the minmax value in the environment of complete information if the contracting game allows principals to use a contract that specifies a menu of actions conditional on the agents' messages. However, it was not clear how to extend his approach to a competing mechanism game with incomplete information. Our paper is the first to extend such an approach to a case with incomplete information. The DIC property can be ensured even without reference to other principals' actions or mechanisms because an agent's payoff, as specified in (1), does not depend on the actions chosen by the principals she does not choose. This makes it possible to assign a menu of DIC direct mechanisms under incomplete information in our framework as if a menu of actions were assigned under complete information in Szentes (2010).

Lastly, Szentes (2010) only allows deterministic actions as Yamashita (2010) does. If a random action is allowed, the minmax and maxmin values in terms of actions are all the same; Even if a contract assigns a menu of actions, instead of a single action, conditional on agents' messages, there are no differences regarding the lower bound. That is not the case here. Let Ω_D be compact, convex. The minmax value in (13) is expressed as

$$\min_{\mu \in \Omega_D} \max_{\mu' \in \Omega_D} \Lambda_J(\mu', \mu), \text{ where } \Lambda_J(\mu', \mu) := \max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi)$$

If Λ_J were to be quasi-concave, upper semi-continuous in μ and quasi-convex, lower semi-continuous in μ' , we could apply Sion's minimax theorem (1958) to swap the min and max. However, it is generally not known whether $\Lambda_J(\mu', \mu)$ satisfies such properties because it is based on a continuation equilibrium played by agents given (μ', μ) .

Let us mention an important implication of Theorem 2. (17) shows that it is sufficient to have a message space H which includes only two messages, because

its role is to reveal whether or not a competing principal deviates. For this, let $H = \{0, 1\}$, $B = \{1\}$, and $B^c = \{0\}$ so that $B \cup B^c = H$. The message 1 implies that the competing principal deviated. The message 0 means that the competing principal did not. Binary messages can be easily adopted in practice (e.g., on-line markets). Let us call g^* with $H = \{0, 1\}$ a *deviation-reporting contract*.

4.1 Φ_J^e versus Φ_J^*

The set of a principal's equilibrium ex-ante expected payoffs $\Phi_J^e = [\underline{\phi}_J, \bar{\phi}_J]$ is clearly a subset of the set of a principal's feasible expected payoffs, $\Phi_J^* = [0, \bar{\phi}_J]$ because a principal's ex-ante expected payoff $\underline{\phi}_J$ upon deviation cannot be less than zero ($\underline{\phi}_J \geq 0$). Whether or not Φ_J^e is equal to Φ_J^* comes down to whether or not non-deviating principals can lower a deviator's payoff $\underline{\phi}_J$ down to his reservation payoff level, which is zero. It depends on whether or not principals have limited liability and also whether or not they have capacity constraint.

If principals (and agents) have no limited liability, so that they can actually transfer positive amounts of money to agents who select them, then any feasible payoff can be supported in an equilibrium.

Theorem 3 *Suppose that there is no limited liability. Then, $\Phi_J^e = \Phi_J^*$.*

Proof. Because both Φ_J^e and Φ_J^* share the same maximum $\bar{\phi}_J$, we only need to show that $\underline{\phi}_J = 0$. That is, a deviating principal's ex-ante expected payoff is simply equal to his reservation payoff. A deviating principal's payoff depends on what direct mechanism non-deviators use in their deviation-reporting contract.

We construct a DIC direct mechanism that non-deviators use to punish a deviator in the following way. Let t^* be an arbitrary amount of money that satisfies

$$\min_{(k,x)} (b^k x + g^k) + t^* \geq 0 \quad (19)$$

$$\max_k (a^k w + y^k) - \max_{(k,x)} (b^k x + g^k) + \min_{(k,x)} (b^k x + g^k) - t^* \leq 0 \quad (20)$$

Suppose that each principal's deviation-reporting contract assigns a DIC direct mechanism, $\mu = \{q^1, \dots, q^K, t\}$, with

$$q^k(\mathbf{x}) = \alpha^k \quad \forall k \in \mathcal{K} \quad (21)$$

$$t(x, \mathbf{x}_{-1}) = t^* \quad \forall (x, \mathbf{x}_{-1}) \in X \times \bar{X}^{N-1}, \quad (22)$$

where $\alpha^k \geq 0$ and $\sum_{k \in \mathcal{K}} \alpha^k \leq 1$. First of all, a direct mechanism satisfying (21) and (22) is clearly DIC since the probability of alternative k being chosen

is constant and the monetary transfer to a participating agent is also constant. Given this DIC direct mechanism, the ex-post payoff for an agent of type x upon participating in the mechanism is

$$\sum_{k \in \mathcal{K}} \alpha^k (b^k x + g^k) + t^* \geq \min_{(k,x)} (b^k x + g^k) + t^* \geq 0$$

Therefore, the individual rationality condition of μ is satisfied for all types of agents. Given the direct mechanism μ with (21) and (22) that each non-deviating principal chooses, suppose that a deviating principal chooses a DIC direct mechanism $\dot{\mu} = \{\dot{q}^1, \dots, \dot{q}^K, \dot{t}\}$. An agent of type x participates in $\dot{\mu}$ only if

$$\dot{T}(x) \geq \sum_{k \in \mathcal{K}} \alpha^k (b^k x + g^k) - \sum_{k \in \mathcal{K}} (b^k x + g^k) \dot{Q}^k(x) + t^*, \quad (23)$$

where the reduced-form mechanism $\{\dot{Q}^1, \dots, \dot{Q}^K, \dot{T}\}$ of $\dot{\mu}$ depends on the agent's selection strategy $\pi(\mu', \mu)$ in a continuation equilibrium. Note that the ex-ante expected transfer from the deviator to an agent is

$$\begin{aligned} & \int_{\underline{x}}^{\bar{x}} \dot{T}(x) dz(\pi(\mu', \mu))(x) \\ & \geq \int_{\underline{x}}^{\bar{x}} \left(\sum_{k \in \mathcal{K}} \alpha^k (b^k x + g^k) - \sum_{k \in \mathcal{K}} (b^k x + g^k) \dot{Q}^k(x) \right) dz(\pi(\mu', \mu))(x) + t^* \\ & \geq \min_{(k,x)} (b^k x + g^k) - \max_{(k,x)} (b^k x + g^k) + t^* \\ & = \max_k (a^k w + y^k) \end{aligned} \quad (24)$$

The first inequality above is satisfied by (23). The second inequality is clear given the definitions of min and max. The equality holds because of (20). Then, the principal's payoff upon deviation to a DIC direct mechanism μ' satisfies

$$\begin{aligned} & \sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \dot{Q}^k(x) dz(\pi(\mu', \mu))(x) - N \int_{\underline{x}}^{\bar{x}} \dot{T}(s_1) dz(\pi(\mu', \mu))(x) \\ & \leq \sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \dot{Q}^k(x) dz(\pi(\mu', \mu))(x) - N \max_k (a^k w + y^k) \leq 0, \end{aligned} \quad (25)$$

where the first inequality comes from (24) and the last inequality is valid because

$$\sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \dot{Q}^k(s_1) dz(\pi(\mu', \mu))(s_1) \leq \max_k (a^k w + y^k)$$

and $\max_k (a^k w + y^k) \geq 0$. (25) implies that a principal cannot get an ex-ante expected payoff greater than zero in any continuation equilibrium upon his deviation to any DIC direct mechanism. This implies that given each non-deviator's DIC direct mechanism μ satisfying (21) and (22), the best response for the deviator is μ_o , defined in (11), which ensures zero payoff. Therefore, $\underline{\phi}_J = 0$ and hence $\Phi_J^e = \Phi_J^*$. ■

Since the DIC direct mechanism μ with (21) and (22) that non-deviating principals use induces $\underline{\phi}_J = 0$ for a deviating principal, it is μ_p that is specified in (18). The key to constructing the DIC direct mechanism μ with (21) and (22) is the fixed monetary transfer t^* to a participating agent, which satisfies (19) and (20). (19) ensures the agent's individual rationality, whereas (20) ensures that the monetary transfer t^* to a participating agent is so high that a deviating principal's ex-ante expected payoff is no greater than his reservation payoff in any continuation equilibrium upon his deviation: t^* may well be positive so that a principal needs to pay.

Suppose that a principal has limited liability so that an amount of monetary transfer to an agent cannot be positive, as assumed in Section 2. For example, sellers may not want to lower their prices below their average cost at least to cover their variable cost in any case. If this is the case, whether Φ_J^e is equal to Φ_J^* depends on whether sellers have capacity constraint. If sellers have no capacity constraints, a deviating seller cannot attract buyers unless his price is below the average cost. Subsequently a deviating seller must incur losses in order to attract a buyer and hence $\underline{\phi}_J = 0$. Therefore, $\Phi_J^e = \Phi_J^*$.

If sellers have capacity constraints, non-deviating sellers may not lower a deviating seller's profit down to zero. For example, assume that each seller can supply at most one unit of the homogeneous good. Even if every non-deviating seller sells his object for free upon a competing seller's deviation, buyers cannot get the object from a non-deviating seller with probability one when there is more than one buyer who selects a non-deviating seller. This implies that a deviating seller's ex-ante expected profit can be positive by selling his object at a price slightly higher than zero: The probability with which buyers visit the deviating seller is lower than the probability with which they visit each non-deviating seller. In this case, generally Φ_J^e is a (strict) subset of Φ_J^* with $\underline{\phi}_J > 0$. Applications in Section 6 focus on sellers with limited liability and with or without capacity constraint.

5 Robustness

A contract defined in the previous section assigns only a menu of DIC direct mechanisms contingent on messages sent by agents. This may restrict a deviating principal's ability to come up with a better mechanism that could provide a higher ex-ante expected payoff. Let $\gamma = \{\sigma^1, \dots, \sigma^K, \tau\}$ be a principal's arbitrary (anonymous) mechanism with a message space $\bar{M} = M \times \{m^\circ\}$ for each agent.¹¹ For all $k \in \mathcal{K}$, $\sigma^k : \bar{M}^N \rightarrow [0, 1]$ specifies the probability of alternative k as the function of agents' messages in $\bar{M}^N$. $\tau : \bar{M}^N \rightarrow \mathbb{R}_-$ specifies monetary transfer to an agent as the function of agents' messages in $\bar{M}^N$. Let Γ be the set of all possible mechanisms with message space $\bar{M}$.

When a competing principal deviates, each non-deviating principal's contract g^* will choose a DIC direct mechanism μ_p given that agents all report that his competitor deviated in a continuation. Suppose that a deviating principal posts a contract that assigns a menu (i.e., subset) of mechanisms in Γ and chooses a mechanism γ from the menu that is assigned. Then, an array of mechanisms (γ, μ_p) defines a subgame played by agents.¹² We can fix truthful type reporting to each non-deviating principal because μ_p is DIC. Then, a continuation equilibrium of the subgame defined by (γ, μ_p) is characterized by the agent's strategy of communicating with the deviating principal, $c(\gamma, \mu_p) : X \rightarrow \Delta(M)$ upon selecting him and her strategy of selecting him, $\pi(\gamma, \mu_p) : X \rightarrow [0, 1]$.

The agent's communication strategy $c(\gamma, \mu_p)$ induces a direct mechanism $\mu_{c, \mu_p}(\gamma) = \{q_{c, \mu_p}^1, \dots, q_{c, \mu_p}^K, t_{c, \mu_p}\}$ from $\gamma = \{\sigma^1, \dots, \sigma^K, \tau\}$. Let I denote the number of agents who select the deviating principal. Then, for every $I \leq N$ and every $(x_1, \dots, x_I) \in X^I$, $\mu_{c, \mu_p}(\gamma) = \{q_{c, \mu_p}^1, \dots, q_{c, \mu_p}^K, t_{c, \mu_p}\}$ is defined as, for all $k \in \mathcal{K}$,

$$q_{c, \mu_p}^k(x_1, \dots, x_I, \mathbf{x}_{-I}^\circ) = \int_M \cdots \int_M \sigma^k(m_1, \dots, m_I, \mathbf{m}_{-I}^\circ) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

$$t_{c, \mu_p}(x_1, \dots, x_I, \mathbf{x}_{-I}^\circ) = \int_M \cdots \int_M \tau^k(m_1, \dots, m_I, \mathbf{m}_{-I}^\circ) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

where $\mathbf{x}_{-I}^\circ = (x^\circ, \dots, x^\circ)$ and $\mathbf{m}_{-I}^\circ = (m^\circ, \dots, m^\circ)$.

¹¹Not participating is equivalent to sending m° .

¹² (γ, μ_p) means that the deviating principal's mechanism is γ , whereas every non-deviating principal's mechanism is μ_p .

The agent's selection strategy $\pi(\gamma, \mu_p)$ induces the probability $z(\pi(\gamma, \mu_p))(x)$ that she either has her type below x as a participant of γ or selects the other principals whose mechanism is μ_p , similar to (3):

$$z(\pi(\gamma, \mu_p))(x) = 1 - \int_x^{\bar{x}} \pi(\gamma, \mu_p)(s) dF. \quad (26)$$

Let $\mathcal{O}$ be the set of all optimal strategies (c, π) for communicating with the deviating principal and selecting a principal in a continuation equilibrium of the subgame defined by (γ, μ_p) for all $\gamma \in \Gamma$. Following (4) and (5), we can derive the reduced-form direct mechanism, $\{Q_{\pi, c, \mu_p}^1, \dots, Q_{\pi, c, \mu_p}^K, T_{\pi, c, \mu_p}\}$ from $\mu_{c, \mu_p}(\gamma) = \{q_{c, \mu_p}^1, \dots, q_{c, \mu_p}^K, t_{c, \mu_p}\}$ and $z(\pi(\gamma, \mu_p))$. Then, it is straightforward to show that $\mu_{c, \mu_p}(\gamma)$ is Bayesian incentive compatible (BIC) for any $(c, \pi) \in \mathcal{O}$: For all $x, x' \in X$,

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) Q_{\pi, c, \mu_p}^k(x) + T_{\pi, c, \mu_p}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) Q_{\pi, c, \mu_p}^k(x') + T_{\pi, c, \mu_p}(x').$$

Note that the Bayesian incentive compatibility of $\mu_{c, \mu_p}(\gamma)$ depends on the mechanism μ_p chosen by the other principals because the agent's communication and selection strategies depend on mechanisms chosen by both principals.

Following (6) and (8), we can then use the reduced-form direct mechanism to derive (i) the interim expected payoff, denoted by $U_J(\mu_{c, \mu_p}(\gamma), \mu_p, \pi, x)$, for the agent of type x associated with selecting the deviating principal with γ and (ii) the ex-ante expected payoff, denoted by $\Phi_J(\mu_{c, \mu_p}(\gamma), \mu_p, \pi)$, for the deviating principal with γ .

Definition 2 *A DIC allocation (μ^*, π^*) that is supported in an equilibrium of a competing mechanism game G is robust if*

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \sup_{\gamma \in \Gamma} \left(\sup_{(c, \pi) \in \mathcal{O}} \Phi_J(\mu_{c, \mu_p}(\gamma), \mu_p, \pi) \right)$$

Definition 2 implies that a principal cannot gain upon his deviation to any arbitrary mechanism in Γ regardless of a continuation equilibrium agents play. Our next theorem establishes the robustness of equilibrium DIC allocations.

Theorem 4 *Any equilibrium (DIC) allocation in a competing mechanism game G is robust and hence any payoff in Φ_j^e is supported by a robust equilibrium allocation.*

We prove Theorem 4 by two steps. We start with some basics. Let Ω be the set of all direct mechanisms. Let $\mathcal{O}_c$ be the projection of $\mathcal{O}$ onto the space of the agent's communication strategies. Then, let $\mathcal{M}(\mu_p)$ be the set of *all incentive compatible direct mechanisms* that can be induced from all mechanisms in Γ that a deviating principal might choose in a continuation equilibrium given the non-deviating principal's DIC direct mechanism $\mu_p \in \Omega_D$:

$$\mathcal{M}(\mu_p) := \left\{ \mu_{c,\mu_p}(\gamma) \in \Omega : \forall c \in \mathcal{O}_c, \forall \gamma \in \Gamma \right\}.$$

We assume that Γ is large enough so that $\mathcal{M}(\mu_p)$ can be embedded into it.

Let $\Pi(\mu, \mu_p)$ be the set of all optimal strategies of selecting a deviating principal in a continuation equilibrium where the deviating principal chooses an incentive compatible direct mechanism μ directly from $\mathcal{M}(\mu_p)$. We can then establish the following lemma.

Lemma 1 *Given $\mu_p \in \Omega_D$, the following equality holds:*

$$\sup_{\gamma \in \Gamma} \left(\sup_{(c,\pi) \in \mathcal{O}} \Phi_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi) \right) = \sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right). \quad (27)$$

Proof. Given μ_p that each non-deviating principal chooses, suppose that the deviating principal's mechanism is $\gamma \in \Gamma$. Given (γ, μ_p) , let each agent communicate with the deviating principal upon selecting him and select a principal according to $c(\gamma, \mu_p)$ and $\pi(\gamma, \mu_p)$ for some $(c, \pi) \in \mathcal{O}$. Since $\mathcal{O}$ is the set of all possible communication and selection strategies in a continuation upon a principal's deviation to a mechanism in Γ , $\mu_{c,\mu_p}(\gamma) \in \mathcal{M}(\mu_p)$ and further it is a continuation equilibrium that each agent uses $\pi(\mu_{c,\mu_p}(\gamma), \mu_p) = \pi(\gamma, \mu_p)$ to select a principal and truthfully report her type to the principal who she selects when the deviating principal directly chooses $\mu_{c,\mu_p}(\gamma)$. If Γ is large enough for $\mathcal{M}(\mu_p)$ to be embedded into it, this directly implies (27). ■

Proof. Lemma 1 implies that a DIC allocation (μ^*, π^*) that is supported in an equilibrium of a competing mechanism game G is robust if and only if

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \quad (28)$$

We can then use the BIC-DIC equivalence to establish the robustness of an equilibrium DIC allocation. ■

Corollary 2 *Suppose that a deviating principal offers $\tilde{\mu} \in \mathcal{M}(\mu_p)$, whereas each non-deviating principal uses μ_p to punish the deviator and that it is a continuation equilibrium that agents choose the deviator according to $\tilde{\pi}(\tilde{\mu}, \mu_p)$ given $(\tilde{\mu}, \mu_p)$. Then, there exists a DIC direct mechanism $\mu \in \Omega_D$ such that given (μ, μ_p) , it is a continuation equilibrium that agents choose the deviator according to $\pi(\mu, \mu_p) = \tilde{\pi}(\tilde{\mu}, \mu_p)$ and*

$$U_J(\mu, \mu_p, \pi, x) = U_J(\tilde{\mu}, \mu_p, \tilde{\pi}, x), \forall x \in X \quad (29)$$

$$\Phi_J(\mu, \mu_p, \pi) = \Phi_J(\tilde{\mu}, \mu_p, \tilde{\pi}). \quad (30)$$

Proof. The proof is almost the same as the proof of Corollary 1. The only difference is that, with respect to the deviating principal's point of view, the probability that either an agent's type is less than x or she selects a non-deviator is calculated based on $\pi(\mu, \mu_p)$ and $\tilde{\pi}(\tilde{\mu}, \mu_p)$, that is, $z(\pi(\mu, \mu_p))(x)$ and $z(\tilde{\pi}(\tilde{\mu}, \mu_p))(x)$. ■

To complete the proof of Theorem 4, first note that (14) and (18) imply that a DIC allocation (μ^*, π^*) that is supported in an equilibrium of a competing mechanism game G satisfies

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \max_{\mu \in \Omega_D} \left(\max_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right)$$

Because $\Omega_D \subset \mathcal{M}(\mu_p)$, it is clear that

$$\sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \geq \max_{\mu \in \Omega_D} \left(\max_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \quad (31)$$

(30) in Corollary 2 means that (31) holds with equality and hence (28) is satisfied for any equilibrium DIC allocation. This implies that any equilibrium allocation of a competing mechanism game G is robust. This statement and Theorem 2 directly imply that any payoff in Φ_J^e is supported by a robust equilibrium allocation. This completes the proof of Theorem 4.

A competing mechanism game G lets non-deviating principals to punish a deviating principal with a DIC direct mechanism. Since there is no loss of generality to focus on DIC allocations by Corollary 1, Theorem 4 implies the following remark:

Remark 1 Φ_J^e is the complete set of a principal's ex-ante payoffs associated with all robust equilibrium allocations that are supportable with punishment carried out by a DIC direct mechanism.

Theorem 3 shows that with no limited liability, Φ_J^e is simply equal to the set of a principal's feasible ex-ante expected payoffs. Applications in the next section study the cases with limited liability.

6 Applications

Each buyer has a unit demand for a product. If she consumes the product and pays p , her utility is $x - p$, where x is her valuation that follows a probability distribution F over $X = [0, 1]$. There are N buyers who are looking for the product. Each buyer's reservation utility is zero. There are J sellers with limited liability such that they cannot transfer a positive amount of money to a buyer. Sellers and buyers are all risk neutral.

6.1 Competing prices

Sellers can produce homogeneous products at a constant marginal cost, normalized to zero, without capacity constraint. Each seller's reservation profit is zero. We first show that even if sellers have limited liability, any feasible ex-ante expected payoff in Φ_J^* can be supported in a robust equilibrium of a competing mechanism game G . Furthermore, those payoffs can be easily supported through a deviation-reporting contract g^* that assigns a single price conditional on the buyer's messages of whether or not a competing seller deviated.

Theorem 5 *Suppose that the hazard rate, $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x .*

1. *The set of a seller's ex-ante expected profit that can be supported in a robust equilibrium of a competing mechanism game G is $\Phi_J^e = \Phi_J^* = [0, \bar{\phi}_J]$ with*

$$\bar{\phi}_J = \frac{N}{J} x^* (1 - F(x^*)), \quad (32)$$

where $x^* = \max \left\{ x : x - \frac{1-F(x)}{f(x)} = 0 \right\}$.

2. *Any $\phi \in \Phi_J^* = [0, \bar{\phi}_J]$ can be supported in a robust equilibrium where each principal posts a deviation-reporting contract g^* that offers a single price contingent on buyer's messages $(h_1, \dots, h_N) \in H^N$ with $H = \{0, 1\}$ such that*

$$g^*(h_1, \dots, h_N) := \begin{cases} 0 & \text{if } |\{i : h_i = 1\}| > N/2, \\ x(\phi) & \text{otherwise,} \end{cases} \quad (33)$$

where $x(\phi)$ satisfies $\phi = \frac{N}{J} x(1 - F(x))$.

Proof. See Appendix B. ■

According to Theorems 2 and 4, $\Phi_J^e = [\underline{\phi}_J, \bar{\phi}_J]$ is the set of a seller's ex-ante expected profits that can be supported in a robust pure-strategy equilibrium of

a competing mechanism game G . We first show that a seller's ex-ante expected profit $\bar{\phi}_J$ in the joint profit maximization is expressed as (32). Second, we can show that any $\phi \in \Phi_J^* = [0, \bar{\phi}_J]$ can be supported in a robust equilibrium with the deviation reporting contract g^* specified in (33).

In the seller's joint profit maximization, all sellers post an identical direct mechanism $\mu = \{q, p\}$ such that mappings $q : X^N \rightarrow [0, 1]$ and $p : X^N \rightarrow \mathbb{R}_+$ specify the probability that a buyer receives the product and her payment to the seller respectively (Note that $X = [0, 1]$ so that, if a buyer does not choose, her type message is regarded as zero and $q(0, \mathbf{x}_{-i}) = p(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$). Because every seller's mechanism is identical, a buyer of any type x selects a seller with $\pi(\mu, \mu)(x) = 1/J$.¹³ For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ . One can then derive a reduced-form direct mechanism $Q : X \rightarrow [0, 1]$ and $P : X \rightarrow \mathbb{R}_+$ similar to (4) and (5), based on $z(\pi)(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ in (9).

Because a seller can produce each unit at a constant cost and each buyer's valuation is i.i.d, the seller's joint profit maximization problem is to find $Q : X \rightarrow [0, 1]$ and $P : X \rightarrow \mathbb{R}_+$ that maximize

$$N \int_0^1 P(x) dz(\pi)(x) \quad (34)$$

subject to the incentive compatibility and individual rationality conditions. Given (9), we can show that the virtual valuation of a buyer with valuation x is

$$x - \frac{1 - z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1 - F(x)}{f(x)}.$$

Given the monotone hazard rate on $x \in [0, 1]$, it is then clear that the product must be sold only to a buyer whose virtual valuation is equal to zero or higher with probability one. A buyer's payment is then

$$P(x) = xQ(x) - \int_0^x Q(s)ds = \begin{cases} x^* & \text{if } x \geq x^*, \\ 0 & \text{otherwise.} \end{cases} \quad (35)$$

Therefore, the seller's ex-ante expected profit in the joint profit maximization is (32), given (34) and (64). Furthermore, any feasible ex-ante profit for a seller $\phi \in \Phi_J^* = [0, \bar{\phi}_J]$ can be expressed as $\phi = \frac{N}{J} x(\phi) (1 - F(x(\phi)))$ for some $x(\phi) \in [0, 1]$ as shown in Appendix B. Therefore, any feasible ex-ante profit for a seller ϕ can be supported in a robust equilibrium where each seller's deviation-reporting contract assigns either zero or $x(\phi)$, depending on buyers' messages on whether or not a competing seller deviates.

¹³If a buyer does not choose any seller, it is equivalent to choosing a seller with equal probability and sending the zero type message.

Theorem 5 shows that the set of a principal's feasible ex-ante expected profits Φ_J^* is the complete set of a principal's ex-ante expected payoffs that are robustly supportable in an equilibrium of a competing mechanism game G despite the seller's limited liability (i.e., he cannot charge a negative price). This is due to there being no capacity constraints: Non-deviating sellers can attract all the buyers in the market at the price equal to their constant marginal cost (zero) and then the deviator cannot gain in any continuation equilibrium upon his deviation to any arbitrary mechanism. Further, the proof of Theorem 5 shows that any payoff ϕ in Φ_J^* can be easily supported by the deviation-reporting contract that offers a single price depending on buyers' binary messages.

6.1.1 Fixed entry cost

Given the number of buyers N , we have assumed that there is a fixed number of sellers in the market. However, we can endogenize the number of sellers in the market. Suppose that a seller has to incur a fixed cost, $C > 0$, to enter the market.

Note that a seller's *gross* ex-ante expected profit $\phi \in \Phi_J^*$ be seen as an equal split of $Nx(1 - F(x))$ as shown in (64). Assume that $\frac{N}{2}x^*(1 - F(x^*)) - C \geq 0$.¹⁴ Recall that $\frac{N}{J}x(1 - F(x))$ is maximized at $x = x^*$ by excluding buyers whose valuation is less than x^* . Given the monotone hazard rate on x , a seller's net ex-ante expected profit $\frac{N}{J}x(1 - F(x)) - C$ is increasing in x before $x = x^*$ and decreasing in x after. Furthermore, it is equal to $-C$ at $x = 0$ or 1 . Therefore, for all J such that

$$\frac{N}{J}x^*(1 - F(x^*)) - C \geq 0,$$

we can then derive the threshold of the price $x_J^- \leq x^*$ and $x_J^+ \geq x^*$ that satisfy

$$\frac{N}{J}x_J^-(1 - F(x_J^-)) - C = \frac{N}{J}x_J^+(1 - F(x_J^+)) - C = 0. \quad (36)$$

Theorem 6 *Given the number of buyers N ,*

1. *the number of sellers in the market in an equilibrium of a competing mechanism game G is unique and equal to $J_c \in \max \{J : x_J^- \leq x^*\}$.*
2. *$\Phi_{J_c}^* = [0, \frac{N}{J_c}\bar{R} - C]$ is the complete set of a principal's ex-ante profits that can be robustly supported in an equilibrium, where $\bar{R} = x^*(1 - F(x^*))$*
3. *$[x_{J_c}^-, x_{J_c}^+]$ is the range of prices that can be supported in an equilibrium of competing mechanism game G*

¹⁴This assumption implies that there will be at least two sellers in the market.

Proof. Because x_J^- is increasing in J , $\max \{J : x_J^- \leq x^*\}$ is a singleton so that we have a unique J_c .¹⁵ Suppose that the number of sellers in the market is $J < J_c$. If one additional seller enters the market, then we have $J + 1 \leq J_c$ and a seller's profit net of fixed cost in the market with $J + 1$ sellers is as low as zero and as high as $\frac{N}{J+1}\bar{R} - C > 0$. Therefore, it is (weakly) dominant for a potential seller to enter the market. We assume that if a seller is indifferent between staying out of the market and entering it, he enters the market. A seller will enter the market as long as $J < J_c$. If $J = J_c$, then $\frac{N}{J+1}\bar{R} - C < 0$. This implies that a seller's profit net of fixed costs in the market with $J + 1$ sellers is negative. No seller will enter the market if the number of sellers in the market is J_c . Therefore, the unique number of firms in the market is J_c .

Let ϕ^C denote a seller's ex-ante expected net profit. If $\phi^C = 0$, $x = x_{J_c}^-$ or x_J^+ according to (36). If $\phi^C = \frac{N}{J_c}\bar{R} - C$, $x = x^*$. Further, $\frac{N}{J_c}x(1 - F(x)) - C$ is continuous and increasing in x before $x = x^*$ and decreasing in x after. Therefore, for any given $\phi^C \in \Phi_{J_c}^*$, there exists a price $x(\phi^C)$ such that

$$\frac{N}{J_c}x(\phi^C)(1 - F(x(\phi^C))) - C = \phi^C.$$

For any $\phi^C \in \Phi_{J_c}^*$, we can then use a deviation-reporting contract defined in (33) to sell the product at price $x(\phi^C)$ on the equilibrium path but at zero price off the path. This also shows that the range of equilibrium prices is $[x_{J_c}^-, x_{J_c}^+]$ ■

The intuition of Theorem 6 is clear. If the number of sellers J is less than J_c , a potential seller finds it always optimal to enter the market because he can ensure that his ex-ante expected net profit will be at least as high as his reservation profit, i.e., $\phi^C \in \Phi_{J_c}^*$. If the number of sellers in the market is $J = J_c$, there are already too many sellers in the sense that if one additional seller enters the market, the maximum ex-ante expected profit associated with monopoly price x^* is negative (i.e., $\frac{N}{J_c+1}\bar{R} - C < 0$). Therefore, the number of sellers in the market is uniquely determined and it is equal to J_c .

While the number of sellers in the market is uniquely determined, their ex-ante expected net profit can be any level in $\Phi_{J_c}^* = \left[0, \frac{N}{J_c}\bar{R} - C\right]$ in the finite market. We are interested in how the ranges of net profits and prices change as the number of buyers increases.

Theorem 7 *As $N \rightarrow \infty$, a seller's equilibrium ex-ante expected profit is uniquely determined and is equal to zero, and the equilibrium price is also uniquely determined and is equal to the monopoly price, x^* .*

¹⁵Equivalently, $J_c \in \min \{J : x_J^+ \geq x^*\}$ is unique because x_J^+ is decreasing in J .

Proof. Let s be the ratio of the number of buyers to the number of sellers, i.e., $s = \frac{N}{J}$. Define the ratio s^* that satisfies

$$s^* \bar{R} - C = 0 \quad (37)$$

Given the number of buyers N , J_c is indeed equal to $\lfloor N/s^* \rfloor$, which denotes the largest integer that does not exceed N/s^* . Therefore the upper bound of a seller's ex-ante expected profit is

$$\frac{N}{\lfloor N/s^* \rfloor} \bar{R} - C \quad (38)$$

Since $\lim_{N \rightarrow \infty} \frac{N}{\lfloor N/s^* \rfloor} = s^*$, the upper bound of a seller's expected profit converges to zero as $N \rightarrow \infty$. Because $x_{J_c} = x_{\lfloor N/s^* \rfloor}$ is the price that makes a seller's ex-ante expected profit equal to zero as shown in (36), this implies that

$$\lim_{N \rightarrow \infty} x_{\lfloor N/s^* \rfloor} = x^*.$$

Because $x_{\lfloor N/s^* \rfloor}$ is the upper bound of the equilibrium price, the monopoly price is the unique equilibrium price as $N \rightarrow \infty$. ■

Example 1 Suppose that a buyer's valuation follows the uniform distribution over $[0, 1]$. Then, the monopoly price is $x^* = 1/2$, which is the highest possible equilibrium price. Let the fixed entry cost $C = 1$. The ratio of the number of buyers to the number of sellers that makes the highest equilibrium price equal to zero is then $s^* = 4$. The following table shows the unique equilibrium number of sellers ($J_c = \lfloor N/s^* \rfloor$), the equilibrium ratio of the number of buyers to the number of sellers ($\frac{N}{\lfloor N/s^* \rfloor}$), the range of a seller's equilibrium ex-ante expected profit ($\Phi_{J_c}^*$), and the range of equilibrium prices ($[x_{J_c}^-, x_{J_c}^+]$), given the number of buyers (N).

N	10	20	70	150	1550
$\lfloor N/s^* \rfloor$	2	5	17	37	387
$\frac{N}{\lfloor N/s^* \rfloor}$	5	4	4.12	4.05	4.005
$\Phi_{J_c}^*$	$[0, 0.25]$	0	$[0, 0.03]$	$[0, 0.01]$	$[0, 0.001]$
$[x_{J_c}^-, x_{J_c}^+]$	$[0.28, 0.72]$	0.5	$[0.42, 0.58]$	$[0.44, 0.56]$	$[0.48, 0.52]$

Table 1. Equilibrium Characterization

For any given number of buyers N , there exists a unique equilibrium number of sellers, which is $\bar{J} = \lfloor s^* N \rfloor$. The actual ratio of the number of buyers to sellers $s = \frac{N}{\lfloor N/s^* \rfloor}$ is not monotone in N . Given $s^* = 4$, the actual ratio is exactly

equal to $1/4$ when there are twenty buyers in the market. Therefore, we can have the unique equilibrium outcome even with a finite number of buyers in that a seller's unique equilibrium profit is zero and the equilibrium price is equal to the monopoly price x^* . What is also clear is that as $N \rightarrow \infty$, the actual ratio of the number of buyers to the number of sellers approaches $s^* = 4$, the upper bound of a seller's equilibrium ex-ante expected profit converges to zero, and the range of equilibrium prices $[x_{J_c}^-, x_{J_c}^+]$ converges to the monopoly price x^* . Note that $x_{J_c}^-$ and $x_{J_c}^+$ can be viewed as the competitive prices because these are the prices at which a seller's ex-ante expected profit is zero. As $N \rightarrow \infty$, these competitive prices converge to the monopoly price and, in the limit, the competitive price is the monopoly price and all sellers make zero profits.

6.2 Competing auctions

In Section 6.1, each seller has no capacity constraint. What if each seller can produce at most a single unit? The literature has studied competing auctions where sellers are restricted to choose their reserve prices in the (second price) auction. This has created challenges in several aspects of modeling. First of all, it is hard to establish the existence of an equilibrium of competing auctions with a finite number of sellers and buyers. Virag (2010) and Burguet and Sákovics (1999) show the existence of a mixed-strategy equilibrium where a seller employs a mixed strategy over reserve prices for his auction. However, a mixed-strategy equilibrium in a competing mechanism game with a restricted set of (DIC) direct mechanisms is generally not robust to the possibility that a seller may deviate to a more complex mechanism.¹⁶

The literature shows that there is a pure-strategy equilibrium of the competing auction game in a large market (Peters 1997, Peters and Severinov 1997) and that it is robust in that no seller can gain by deviating to any arbitrary mechanism (Han 2015). In this equilibrium, every seller sets reserve price equal to his cost of producing the product. While the literature has found one robust equilibrium, it is not yet known if there are additional robust equilibria.

Theorems 2 and 4 show that $\Phi_J^e = [\underline{\phi}_J, \bar{\phi}_J]$ is the set of a principal's robust equilibrium ex-ante expected payoffs in a competing mechanism game G . The next theorem identifies $\bar{\phi}_J$ when sellers can produce at most a single unit at the constant cost of zero.

Theorem 8 *Suppose that the hazard rate $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x . Then $\bar{\phi}_J$ is reached when all sellers offer an auction with reserve price x^* such*

¹⁶This is because a seller can deviate to a more complex mechanism where he sets his reserve price contingent on buyers' messages about reserve prices set by his competing sellers.

that

$$x^* \in \max \left\{ x : x - \frac{1 - F(x)}{f(x)} = 0 \right\} \quad (39)$$

Each seller's ex-ante expected profit is

$$\bar{\phi}_J = \frac{N}{J} \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) \left(1 - \frac{1 - F(x)}{J} \right)^{N-1} f(x) dx. \quad (40)$$

Let s be the ratio of the number of buyers to the number of sellers, i.e., $s = \frac{N}{J}$. In a large market, it becomes

$$\bar{\phi}_\infty := \lim_{J \rightarrow \infty} \bar{\phi}_J = s \int_{x^*}^1 \left[\left(x - \frac{1 - F(x)}{f(x)} \right) e^{-s(1-F(x))} \right] f(x) dx. \quad (41)$$

Proof. See Appendix C. ■

Because a seller has at most a single unit to sell, it is important whom he should sell it to among those buyers who select him. $\bar{\phi}_J$ is a seller's ex-ante expected profit in the joint profit maximization where every seller uses an identical selling mechanism. Therefore, each buyer with valuation x selects a seller with equal probability $\pi(\mu, \mu)(x) = 1/J$ and hence the probability that a buyer whose valuation is less than x or she selects another seller is given by

$$z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x.$$

Then, a buyer's virtual valuation is

$$x - \frac{1 - z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1 - F(x)}{f(x)}.$$

Given the monotone hazard rate, the virtual valuation is increasing and the optimal mechanism takes the form of an auction with reserve price: the object goes to the highest valuation buyer if it is sold at all. It is sold if and only if the highest valuation among participating buyers is no less than x^* . Appendix C shows that (40) and (41) are a seller's ex-ante expected profits in finite and large markets respectively.

The lower bound of a seller's robust equilibrium ex-ante expected profit $\underline{\phi}_J$ was simply his reservation profit when there is no limited liability or with no capacity constraint as shown in Sections 4 and 6.1. However, it is very difficult to derive $\underline{\phi}_J$ when there are both limited liability and capacity constraint as in the auction environment. Here, we rather focus on the set of a seller's ex-ante expected profits that can be supported in a robust equilibrium where

non-deviating sellers punish the deviator by changing the reserve price in their auctions. The lower bound of a seller's ex-ante expected profits is then expressed in terms of the minmax value with respect to reserve prices in the auctions in a large market (and also in a finite market when some properties on the probability distribution on a buyer's valuation are satisfied).

6.2.1 Many sellers and many buyers

Let $\mu(x)$ denote an auction with reserve price. Then, $\Phi_J(\mu(x'), \mu(x), \pi)$ denote a seller's expected profit when he posts an auction with reserve price x' given that $J - 1$ sellers all post auctions with reserve price x and buyers select a seller according to their selection strategy π .

The difficulty in a finite market is that if one seller deviates, it affects the pay-offs that buyers can get from selecting a non-deviating seller. This is because a seller's deviation necessarily changes the probabilities with which buyers choose a non-deviating seller. In a large market with infinitely many sellers and buyers, a single seller's mechanism has no impact on the market payoff that a buyer can expect. This makes a deviating seller choose his selling mechanism, taking each buyer's market payoff that she receives from selecting a non-deviating seller as given.

Peters (1997) shows that in a large market, a deviating seller cannot do better with any arbitrary direct mechanism than she does with an auction, given any distribution of reserve prices of auctions chosen by non-deviating sellers. We can apply the result in Han (2015) to show that in a large market, a deviating seller cannot do better with any arbitrary mechanism than he does with a direct mechanism, given any distribution of reserve prices of auctions chosen by non-deviating sellers. Therefore, we have that, for all $x \in [0, 1]$,¹⁷

$$\lim_{J \rightarrow \infty} \left[\sup_{\gamma \in \Gamma} \left(\sup_{(c, \pi) \in \mathcal{O}} \Phi_J(\gamma, \mu(x), c, \pi) \right) \right] = \lim_{J \rightarrow \infty} \left[\max_{x' \in [0, 1]} \Phi_J(\mu(x'), \mu(x), \pi) \right]. \quad (42)$$

This implies that a deviating seller only needs to consider an auction with reserve price even if he can offer any arbitrary selling mechanism. Importantly, (42) holds independent of the non-decreasing property of the buyer's virtual valuation. This is the key to the robustness in a large market. Let

$$\tilde{\phi}_\infty := \lim_{J \rightarrow \infty} \left[\min_{x \in [0, 1]} \left(\max_{x' \in [0, 1]} \Phi_J(\mu(x'), \mu(x), \pi) \right) \right].$$

¹⁷For any given array of auctions offered by sellers, there exists a unique selection strategy π in a continuation equilibrium (Peters and Severinov 1997, Peters 1007, Virag 2010). This is why the maximum over the selection strategies is not taken.

Theorem 9 $\tilde{\Phi}_\infty = [\tilde{\phi}_\infty, \bar{\phi}^\infty]$ is a set of a seller's ex-ante expected profits that can be supported in a robust equilibrium of a competing mechanism game G .

Proof. Let $x^p \in [0, 1]$ be a reserve price such that

$$\lim_{J \rightarrow \infty} \left[\max_{x' \in [0, 1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \tilde{\phi}_\infty.$$

Then, we have

$$\lim_{J \rightarrow \infty} \Phi_J(\mu(x^p), \mu(x^p), \pi) \leq \lim_{J \rightarrow \infty} \left[\max_{x' \in [0, 1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \tilde{\phi}_\infty \quad (43)$$

by the definition of the maximum operator. On the other hand, we have that

$$\lim_{J \rightarrow \infty} \Phi_J(\mu(x^*), \mu(x^*), \pi) = \bar{\phi}^\infty \geq \lim_{J \rightarrow \infty} \left[\max_{x' \in [0, 1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \tilde{\phi}_\infty \quad (44)$$

Furthermore, when every seller's reserve price is the same, a seller's ex-ante expected profit,

$$\lim_{J \rightarrow \infty} \Phi_J(\mu(x), \mu(x), \pi) = s \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1 - F(x'))} \right] f(x') dx',$$

is continuous in x . Given the continuity of $\lim_{J \rightarrow \infty} [\Phi_J(\mu(x), \mu(x), \pi)]$, (43) and (44) imply that there exists at least one x such that

$$\lim_{J \rightarrow \infty} \Phi_J(\mu(x), \mu(x), \pi) = \tilde{\phi}_\infty.$$

Furthermore, given the non-decreasing hazard rate and the definition of x^* , (a) $\lim_{J \rightarrow \infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-increasing in x over $[x^*, 1]$ while it reaches its maximum at $x = x^*$ and its minimum, $\lim_{J \rightarrow \infty} \Phi_J(\mu(1), \mu(1), \pi) = 0$ at $x = 1$ in the range of $[x^*, 1]$, and (b) $\lim_{J \rightarrow \infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-decreasing in x over $[0, x^*]$, while it reaches its maximum at $x = x^*$ and its minimum at $x = 0$ in the range of $[0, x^*]$. Because $\lim_{J \rightarrow \infty} \Phi_J(\mu(x), \mu(x), \pi)$ is continuous in x , (a) and (b) imply that there exists at least one $x(\phi)$ such that $\lim_{J \rightarrow \infty} \Phi_J(\mu(x(\phi)), \mu(x(\phi)), \pi) = \phi$ for all $\phi \in \tilde{\Phi}_\infty$.

For any given $\phi \in \tilde{\Phi}_\infty$, each seller posts the deviation-reporting contract

$$g^*(h_1, \dots, h_N) := \begin{cases} \mu(x^p) & \text{if } |\{i : h_i = 1\}| \geq \frac{N}{2} \\ \mu(x(\phi)) & \text{otherwise} \end{cases}. \quad (45)$$

Given this contract, buyers truthfully report whether or not the competing seller deviates to all sellers, and also their true valuation upon selecting a seller.

Figure 1: Seller's Profits

Because $\phi \geq \tilde{\phi}_\infty$, (42) implies that a seller cannot gain by deviating to any arbitrary mechanism in a large market. ■

For any ex-ante expected profit level ϕ , we can identify at least one reserve price $x(\phi)$ that induces each seller's ex-ante expected profit equal to ϕ when every seller offers an auction with $x(\phi)$. It is then straightforward to maintain the profit level ϕ through the deviation-reporting contract that makes the reserve price as a function on reports by buyers on whether or not a competing seller deviates, as specified in (45).

Example 2 *Each buyer's valuation follows the uniform distribution on $[0, 1]$, and the sellers' cost is zero. Assume that $s = 1$.*

Given reserve price r chosen by all non-deviating sellers, suppose that a deviating seller's reserve price is r' . Peters and Severinov (1997) characterizes the unique continuation equilibrium for any given (r', r) in both a finite market and its limit given a fixed ratio of buyers to sellers.¹⁸ Following their characterization

¹⁸If $r' > r$, then there exists $y' \geq r'$ that leads to a unique continuation equilibrium in which a buyer with valuation y' or higher select every seller with equal probability but a buyer with valuation $x \in [y', r]$ selects every non-deviating seller with equal probability. If $r' < r$, then a buyer with valuation y' or higher selects every seller with equal probability but a buyer with

of the unique continuation equilibrium in Lemma 3 and Theorem 4, our numerical exercise shows that $\tilde{\phi}_\infty \approx 0.104$ is reached when the deviating seller's reserve price is zero given that non-deviating sellers choose zero as their reserve price.¹⁹ This actually happens to the pure-strategy equilibrium of competing auctions in a large market (Peters and Severinov (1997)) and hence the minmax punishment based on reserve prices coincides with the Nash reversion punishment. Because the uniform distribution satisfies the non-decreasing hazard rate, it is easy to see that the seller's profit $\bar{\phi}^\infty \approx 0.213$ in the joint profit maximization is reached when every seller sets his reserve price at $1/2$.

Therefore, any profit level in $[0.104, 0.213]$ is sustainable in equilibrium. Figure 1 shows the seller's ex-ante expected profit when every seller sets an identical reserve price. The line in the middle represents $\tilde{\phi}_\infty \approx 0.104$. The seller's ex-ante expected profit starts from 0.104 at zero reserve price and intersects the line in the middle when the reserve price is 0.874. Therefore, any reserve price in $[0, 0.874]$ can be supported in equilibrium and for any profit level ϕ in $[0.104, 0.213]$, there exists two reserve prices that support ϕ except for $\phi = 0.213$ that is uniquely induced by the reserve price $1/2$.

Fixed entry cost Suppose that given the number of buyers, any seller can enter the market but has to incur a fixed entry cost of C . We can then endogenize the equilibrium ratio of the number of buyers to the number of sellers.

Proposition 1 *The equilibrium selling mechanism is the auction with reserve price x^* . The seller's equilibrium net ex-ante expected profit is zero in that the equilibrium ratio of the number of buyers to the number of sellers and reserve price s^* satisfies*

$$s^* \in \left\{ s : s \int_{x^*}^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x') dx' - C = 0 \right\}. \quad (46)$$

Proof. If s induces a negative ex-ante expected profit net of the fixed cost at reserve price x^* , i.e.,

$$s \int_{x^*}^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x') dx' - C < 0$$

valuation $x \in [y', r']$ chooses only the deviating seller. If $r = r'$, then a buyer with valuation r or higher chooses every seller with equal probability.

¹⁹We also derived the minmax value in a finite market. It is equal to 0.103615 with $N = 1000$ and $J = 1000$ and 0.103636 with $N = 10000$ and $J = 10000$. Therefore, we can see that as the number of buyers and sellers increases, holding $s = 1$, the minmax value approaches its limit value, $\tilde{\phi}_\infty = 0.103638$. It is also the same for the upper bound of a seller's equilibrium ex-ante profit and the range of equilibrium reserve prices.

there is no equilibrium where the seller's net ex-ante expected profit is non-negative. This is because the seller's net ex-ante expected profit is jointly maximized by auctions with reserve price x^* . This implies that sellers in the market will leave the market, so that it is not an equilibrium.

Suppose that s induces a positive ex-ante expected profit net of the fixed cost at reserve price x^* , i.e.,

$$s \int_{x^*}^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x') dx' - C > 0. \quad (47)$$

Because the left hand of (47) is continuous in s , there exists $\epsilon > 0$ such that

$$(s - \epsilon) \int_{x^*}^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-(s-\epsilon)(1-F(x'))} \right] f(x') dx' - C > 0$$

Let $s' = s - \epsilon$ given arbitrary ϵ that satisfies the inequality above. Given $s' < s$, we construct all possible equilibria with a deviation-reporting contract ($s' < s$ implies that there are now more sellers given the number of buyers). Note that given the non-decreasing hazard rate, the seller's ex-ante expected profit,

$$s' \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s'(1-F(x'))} \right] f(x') dx',$$

is non-decreasing in x over $[0, x^*]$ and non-increasing in x over $[x^*, 1]$. Furthermore, the seller's ex-ante expected profit is zero at $x = 1$. This implies that there exists at least one x in $(x^*, 1)$ such that s'

$$s' \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s'(1-F(x'))} \right] f(x') dx' - C = 0$$

Let $X(s')$ be the set of reserve prices that make a seller's ex-ante expected profit net of the fixed cost equal to zero:

$$X(s') := \left\{ x : s' \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s'(1-F(x'))} \right] f(x') dx' - C = 0 \right\} \quad (48)$$

Define $x^+(s') := \max X(s')$ and

$$x^-(s') := \begin{cases} 0 & \text{if } s' \int_0^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s'(1-F(x'))} \right] f(x') dx' - C \geq 0 \\ \min X(s') & \text{otherwise} \end{cases}$$

Whether $x^-(s') = 0$ or not depends on the seller's net ex-ante expected profit associated with zero reserve price. If it is not negative, $x^-(s') = 0$ because one

Figure 2: Ranges of Eq. Profits and Reserve Prices

cannot further lower the reserve price. If it is positive, then $x^-(s')$ will be higher than zero. Then, the complete set of reserve prices that induce non-negative ex-ante expected profit net of the fixed cost is $[x^-(s'), x^+(s')]$ and hence, given the new ratio of the number of buyers to sellers, s' , every seller in the market has a non-negative ex-ante expected net profit in any equilibrium. This implies that s^* must satisfy (46) and the reserve price must be x^* . Once s^* satisfies (46), no seller has an incentive to leave or enter the market. ■

Example 3 *Each buyer's valuation independently follows the uniform distribution on $[0, 1]$ and each seller can produce at most one unit of the good at zero cost: However, a seller has to incur the fixed cost of 0.3 to enter the market.*

We numerically derive the minmax value of a seller's ex-ante expected profit at all possible s and it is reached when the seller sets his price equal to zero given that the other sellers' reserve prices are zero. This is true in both the infinite case and the finite case with a large number of sellers and buyers: For the finite case, we fix the number of buyers to 10,000 and we start the number of sellers at 100 and increase it by 100 to vary s . This minmax value of a seller's ex-ante expected profit is the seller's maximum profit, after he enters the market, that he can get upon his unilateral deviation given others' deviation-reporting contract.

Figure 2 shows the numerical results for the ranges of equilibrium reserve prices and ex-ante profit net of the fixed cost as we increase $1/s$ (i.e., the ratio

of the number of sellers to the number of buyers).²⁰ The minmax value of a seller's ex-ante expected profit net of the fixed cost given $1/s$ is reached when his reserve price is equal to zero given that other sellers' reserve prices are equal to zero.

The left panel shows the range of equilibrium net profits. The upper bound is a seller's jointly maximized profit at the reserve price, 0.5. The lower bound is the maximum between zero and the minmax value of a seller's profit net of the fixed cost. When $1/s$ is very low (i.e, there is a very small number of sellers relative to the number of buyers), the minmax value of a seller's profit net of the fixed cost is very high even though every seller's reserve price is zero. This is because each seller is expected to have a buyer with valuation very close to zero. This is why the lower bound and the upper bound of the equilibrium net profit are virtually identical at $1/s = 0.02$. The net minmax value starts decreasing as $1/s$ increases from $1/s = 0.02$. When $1/s \approx 0.457$, the net minmax value is exactly equal to zero. As $1/s$ further increases, the net minmax value becomes negative so that the lower bound of a seller's equilibrium net profit is supported with only a reserve price higher than zero. The range of equilibrium profits shrinks and eventually collapses to zero at $1/s^* \approx 0.657$ (i.e., $s^* \approx 1.523$),²¹ where every seller's reserve price is the monopoly reserve price, 0.5.

The right panel shows the ranges of equilibrium reserve prices. The horizontal dotted line corresponds to the reserve price equal to 0.5, which is the level of reserve price that jointly maximizes each seller's ex-ante expected profit. The curve above the dotted line shows the upper bound of the reserve prices and the one below shows the lower bound of the reserve prices. Any reserve prices between the two are supportable in equilibrium. For the lower bound, note that, after $1/s \approx 0.457$, the net minmax value of a seller's ex-ante expected profit is negative when every seller's reserve price is zero. Therefore, every seller's reserve price must be positive to have zero net profits as the equilibrium net profit. This is why the lower bound is flat before $1/s \approx 0.457$ and keeps increasing after $1/s \approx 0.457$. The range of the equilibrium reserve prices eventually converges to the monopoly reserve price, 0.5, as $1/s$ approaches 0.657.

²⁰We derive the ranges of equilibrium reserve prices and ex-ante expected net profits in both (a) the finite market with $N = 10000$ and J being increased by 100 from the initial number 100 and (b) its infinite case with the corresponding s . The results are virtually identical.

²¹To be exact, a seller's ex-ante expected profit is 0.00002536 when every seller chooses the monopoly reserve price 0.5 given $N = 10000$ and $J = 6566$. One more seller makes a seller's ex-ante expected profit negative even when every seller chooses the monopoly reserve price 0.5.

6.2.2 Two sellers and two buyers

Virag (2007) studies the robustness of a pure-strategy equilibrium of competing auctions with two sellers and two buyers. Proposition 11 in his paper shows that if the probability density function of the buyer's valuation f , and $x + \frac{F}{f}$ are both increasing, the best response of a seller is to post an auction for any auction posted by the competing seller even when he can post any arbitrary selling mechanism. Since the buyer's valuation belongs to $[0, 1]$ and a seller has limited liability, the set of possible reserve prices is $[0, 1]$.

Let us explain how to apply Proposition 11 in Virag (2010) to generate a set of robust equilibrium allocations and the corresponding ex-ante expected profits for a seller. Our deviation-reporting contract is based on three buyers (agents): Suppose that all buyers truthfully reveal whether or not a competing seller deviates. Even when one agent unilaterally deviates from truthful reporting, the majority of agents still truthfully report so that one agent's deviation does not change the direct mechanism that the non-deviating seller's DDM implements.

If there are only two buyers, one buyer's unilateral deviation creates ties. In this case, a seller does not know which agent reports truthfully. Then a seller can punish both of them with an auction with reserve price equal to one. Adding this rule of punishing both buyers in the case of tie, a seller's deviation-reporting contract can still induce both buyers' truthful reporting. Suppose that a seller punishes a competing seller's deviation with an auction with reserve price, that is, punishment is taken only from auctions with reserve price in $[0, 1]$. Given Proposition 11 in Virag (2010), Lemma 1 implies that²²

$$\begin{aligned} \tilde{\phi}_2 := \min_{x \in [0,1]} \left[\sup_{\gamma \in \Gamma} \left(\sup_{(c,\pi) \in \mathcal{O}} \Phi_2(\gamma, \mu(x), c, \pi) \right) \right] = \\ \min_{x \in [0,1]} \left[\max_{x' \in [0,1]} \Phi_2(\mu(x'), \mu(x), \pi) \right]. \quad (49) \end{aligned}$$

Theorem 10 $\tilde{\Phi}_2 = [\tilde{\phi}_2, \bar{\phi}^2]$ is a set of a seller's ex-ante expected profits that can be supported in a robust equilibrium of a competing mechanism game G .

Proof. Let us redefine x^p as

$$x^p := \arg \min_{x \in [0,1]} \left[\max_{x' \in [0,1]} \Phi_2(\mu(x'), \mu(x), \pi) \right].$$

²²There exists a unique selection strategy π for buyers in a continuation equilibrium given any array of auctions (See Burguet and Sákóvics (1999) and Virag (2010)). This is why the expression after the equality does not have the "max" operator over all possible selection strategies in a continuation equilibrium given an array of auction.

It is clear that

$$\Phi_2(\mu(x^p), \mu(x^p), \pi) \leq \max_{x' \in [0,1]} \Phi_J(\mu(x'), \mu(x^p), \pi) = \tilde{\phi}_\infty \quad (50)$$

by the definition of the maximum operator. On the other hand, we have that

$$\Phi_2(\mu(x^*), \mu(x^*), \pi) = \bar{\phi}^\infty \geq \max_{x' \in [0,1]} \Phi_2(\mu(x'), \mu(x^p), \pi) = \tilde{\phi}_\infty \quad (51)$$

Furthermore, when every seller's reserve price is the same, a seller's ex-ante expected profit,

$$\Phi_2(\mu(x), \mu(x), \pi) = \int_x^1 \left(s - \frac{1 - F(s)}{f(s)} \right) \left(1 - \frac{1 - F(s)}{J} \right)^{N-1} f(s) ds,$$

is continuous in x . Given the continuity of $\Phi_2(\mu(x), \mu(x), \pi)$, (50) and (51) imply that there exists at least one x such that

$$\Phi_2(\mu(x), \mu(x), \pi) = \tilde{\phi}_\infty.$$

Furthermore, given the non-decreasing hazard rate and the definition of x^* , (a) $\Phi_2(\mu(x), \mu(x), \pi)$ is non-increasing in x over $[x^*, 1]$ while it reaches its maximum at $x = x^*$ and its minimum, $\Phi_2(\mu(1), \mu(1), \pi) = 0$ at $x = 0$ in the range of $[x^*, 1]$, and (b) $\Phi_2(\mu(x), \mu(x), \pi)$ is non-decreasing in x over $[0, x^*]$ while it reaches its maximum at $x = x^*$ and its minimum at $x = 0$ in the range of $[0, x^*]$. Because $\Phi_2(\mu(x), \mu(x), \pi)$ is continuous in x , (a) and (b) imply that there exists at least one $x(\phi)$ such that $\Phi_J(\mu(x(\phi)), \mu(x(\phi)), \pi) = \phi$ for all $\phi \in \tilde{\Phi}_2$.

For any given $\phi \in \tilde{\Phi}_2$, each seller posts the deviation-reporting contract

$$g^*(h_1, h_2) := \begin{cases} \mu(x^p) & \text{if } |\{i : h_i = 1\}| = 2, \\ \mu(x(\phi)) & \text{if } |\{i : h_i = 0\}| = 2, \\ \mu(1) & \text{otherwise} \end{cases}.$$

Given this contract, buyers truthfully report whether or not the competing seller deviates to all sellers, and also their true valuation upon selecting a seller. Because $\Phi_2(\mu(x), \mu(x), \pi) \geq \tilde{\phi}_2$, (49) implies that a seller cannot gain by deviating to any arbitrary mechanism. Therefore, sellers can implement any auction allocation $(\mu(x), \pi) \in Z$ in a robust equilibrium. ■

Theorem 12 in Virag (2007) shows that if f and $x + \frac{F}{f}$ are both increasing and $0 \geq \frac{1}{2f(0)}$, then there exists a pure-strategy equilibrium of competing auctions where each seller holds an auction with reserve price $r = 0$ and it is robust. The condition $0 \geq \frac{1}{2f(0)}$ however seems quite restrictive. It is not satisfied in Example 4 below; Nonetheless, there exists a range of reserve prices for auctions that can be supported in a robust equilibrium of a competing mechanism game G .

Example 4 Suppose that each buyer's valuation follows a probability distribution $F(x) = x^2$ on $[0, 1]$.

Note that the density $f(x) = 2x$ and $x + \frac{F(x)}{f(x)}$ are both increasing in x so that it satisfies the properties in Virag (2007), which ensures that the best response of a seller is to post an auction for any auction posted by the competing seller even when he can post any arbitrary selling mechanism. Our numerical exercise shows that the minmax value of a seller's ex-ante expected profit, $\tilde{\phi}_2 \approx 0.171$, is achieved when the seller's reserve price is 0.262 when the other sellers' reserve prices are zero: In the finite market a deviating seller's best-response reserve price is not zero when non-deviating sellers' reserve prices are zero.

$\tilde{\phi}_2 \approx 0.171$ is the lower bound of a seller's equilibrium ex-ante expected profit and is achieved when everyone's reserve price is 0.075 or 0.894. The upper bound $\bar{\phi}^2$ is associated with the joint profit maximization, which is achieved when every seller's reserve price is 0.577 and $\bar{\phi}^2 \approx 0.385$. Therefore, the range of a seller's equilibrium profit is $\tilde{\Phi}_2 = [0.171, 0.577]$ and the range of equilibrium reserve prices is $[0.075, 0.894]$.

7 Conclusion

It is quite difficult to develop a tractable competition model that is robust to the possibility that sellers might deviate to an arbitrary selling mechanism due to the complexity of communication on market information. It is not surprising to see that the contributions of the literature on competing mechanisms largely remain in the realm of pure theory in terms of proposing mechanisms that ensure robust equilibrium allocations and the threshold of a seller's equilibrium payoff with respect to such mechanisms (Epstein and Peters 1999, Yamashita 2010, Peters and Troncoso-Valverde 2013, etc.). Szentes (2010) suggests such theoretical contributions are based on a controversial assumption that a seller's mechanism delegates his action (or terms of trade) completely to buyers in the sense that an array of messages sent by buyers determines his action. Szentes (2010) shows that if an array of messages sent by buyers rather selects a menu of actions from which a seller subsequently chooses his favorable action, the threshold of a seller's equilibrium allocation rises to the minmax value with respect to actions from the maxmin in the case of complete information. However, it is not clear how to apply his approach to the case of incomplete information.

Our paper extends the idea of Szentes (2010) to the case incomplete information by allowing a seller's contract to specify a menu of DIC direct mechanisms as a function of an array of messages sent by buyers, and a seller to choose

his preferred DIC direct mechanism from the menu. This leads us to characterize the set of a seller's profits supportable in a (symmetric) equilibrium as a connected interval between the minmax value of the seller's profit with respect to DIC direct mechanisms and the profit in the joint profit maximization. In the standard environment with linear utilities and independent private types, this set of a seller's equilibrium profits is robust to the possibility of a seller's deviation to any arbitrary selling mechanism. In any equilibrium, it is enough for a seller to use a deviation-reporting contract that assigns a single DIC direct mechanism as a function of an array of binary messages sent by buyers. This provides a great deal of convenience since most information that can be collected in on-line markets is a collection of binary messages.

The set of a seller's (robust) equilibrium profits coincides with the set of his feasible (i.e., individually rational and incentive compatible) profits with no limited liability or with no capacity constraints. If sellers have limited liability and capacity constraints (e.g., a seller can produce at most one unit), the lower bound of a seller's equilibrium profit is usually higher than the lower bound of his feasible profit (i.e., reservation profit). While it is hard to identify the lower bound of a seller's equilibrium profit, we show that a seller's equilibrium profit can be as low as the minmax value of his profit with respect to reserve prices of auctions if non-deviating sellers punish a deviator by changing reserve prices in their auctions. If sellers have no capacity constraints, the lower bound of a seller's equilibrium profit is his reservation profit and sellers can sustain any equilibrium profit if they punish a deviator by changing their prices.

This analysis is based on a fixed number of sellers and buyers. One may think of this as the short-run equilibrium analysis where sellers cannot exit the market immediately and, rather, they produce as long as the price is high enough to cover the average variable cost. Sellers would be free to leave or enter the market in the long run. We endogenize the number of sellers for the long-run equilibrium analysis. It is shown that the number of sellers is equal to the largest number at which a seller's equilibrium profit associated with the joint profit maximization is non-negative. As the number of buyers increases, the range of a seller's profit shrinks to zero and only the monopoly terms of trade prevails in equilibrium. Therefore, competition is neutralized in a large market and the beneficial effect of competition is absent.

Our approach can be used in various applications for the directed search model such as competing prices, competing auctions and competition in on-line markets. We believe that our approach can also be applied to other problems. For example, Norman (2004) considers public good provision with exclusion for a single mechanism designer. Using the results from this paper we can also consider competing public good provisions with exclusion where buyers

eventually select one seller for a public good.

This paper is based on the private value environment in the sense that each agent's type affects only her payoffs. It is not difficult to imagine an interdependent value environment where an agent's payoff depends on other agents' types as well. In this case, one can consider a set of ex-post incentive compatible direct mechanisms (Bergemann and Morris 2005) to punish a deviator. The property of ex-post incentive compatibility also does not depend on the endogenous distribution of the number of participating agents given that the participating agents report their true types. Therefore, one can always fix truthful type reporting to non-deviators.

Appendix A. Proof of Corollary 1

Consider any feasible allocation $(\tilde{\mu}, \tilde{\pi}) \in Z$. Therefore, it is a continuation equilibrium that each agent of type x chooses each principal with equal probability $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) = 1/J$ whenever $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) > 0$ and reports her true type upon selecting a principal. Then, a principal's ex-ante expected payoff and an agent's interim expected payoff, $\Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi})$ and $U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x)$, are derived with the type distribution $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))(x)$.

Let $\tilde{Q}^k$ be the reduced form of $\tilde{q}^k$ in $\tilde{\mu} = \{\tilde{q}^1, \dots, \tilde{q}^K, \tilde{t}\}$ based on $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))$. Let Q^k be the reduced form of q^k in $\mu = \{q^1, \dots, q^K, t\}$ based on $z(\pi(\mu, \mu))$ with $\pi(\mu, \mu) = \tilde{\pi}(\tilde{\mu}, \tilde{\mu})$. Then, we define $\tilde{V}(x) := \sum_{k \in \mathcal{K}} b^k \tilde{Q}^k(x)$ and $V(x) := \sum_{k \in \mathcal{K}} b^k Q^k(x)$. If $\pi(\mu, \mu) = \tilde{\pi}(\tilde{\mu}, \tilde{\mu})$, then

$$z(\pi(\mu, \mu)) = z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu})) \quad (52)$$

Because $z(\pi(\mu, \mu))$ is a probability distribution over $\bar{X}$, and $\bar{X}$ is the message space for an agent in a direct mechanism, (52) implies that the probability distribution over $\bar{X}$ for a direct mechanism is preserved and that they are independent of each other. Let $\mathbb{E}[\cdot | z(\pi(\mu, \mu))]$ be the expectation operator over an agent's type x given the probability distribution $z(\pi(\mu, \mu))$. Given the linear payoff structure, we can apply Theorem 1 and Lemma 3 in Gershkov, et al. (2013) for the case of anonymous (and hence non-discriminatory) mechanisms to show the existence of a DIC direct mechanism μ such that

$$V(x) = \tilde{V}(x) \text{ for all } x, \quad (53)$$

$$\mathbb{E}[Q^k(x) | z(\pi(\mu, \mu))] = \mathbb{E}[\tilde{Q}^k(x) | z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \text{ for all } k \in \mathcal{K} \quad (54)$$

with the transfers t , which preserves each agent's interim expected payoff upon selecting the principal. That is,

$$U_J(\mu, \mu, \pi, x) = U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x) \text{ for all } x \quad (55)$$

Note that (54) is the “*ex-ante*” probability that alternative k is chosen. Taking the expected value of each side of (55) over x and applying (53) yields

$$\begin{aligned}
& \mathbb{E}[T(x)|z(\pi(\mu, \mu))] \\
&= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \\
&+ g^k \left(\sum_{k \in \mathcal{K}} \mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] - \sum_{k \in \mathcal{K}} \mathbb{E}[Q^k(x)|z(\pi(\mu, \mu))] \right) \\
&= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))]
\end{aligned} \tag{56}$$

The first and the second equalities in (56) hold because of (53) and (54) respectively. On the other hand, the principal’s ex-ante expected payoff associated with μ satisfies

$$\begin{aligned}
& \Phi_J(\mu, \mu, \pi) \\
&= \sum_{k \in \mathcal{K}} (a^k w + y^k) \mathbb{E}[Q^k(x)|z(\pi(\mu, \mu))] - N \times \mathbb{E}[T(x)|z(\pi(\mu, \mu))] \\
&= \sum_{k \in \mathcal{K}} (a^k w + y^k) \mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] - N \times \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \\
&= \Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi})
\end{aligned} \tag{57}$$

where the second equality holds because of (54) and (56). Therefore, the principal’s ex-ante expected payoff is preserved. For any ϕ in Φ_J^* , we can find a (BIC) allocation $(\tilde{\mu}, \tilde{\pi})$ that supports it. (57) implies that it can be supported by a DIC allocation, i.e., $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$.

Because monetary transfer to a participating agent is restricted to be non-positive, we need to show whether a DIC direct mechanism μ also has the property of “non-positive” monetary transfer to a participating agent given that the original BIC direct mechanism $\tilde{\mu}$ has that property. Note that given $\tilde{\mu} = \{\tilde{q}^1, \dots, \tilde{q}^K, \tilde{t}\}$, we have

$$\tilde{T}(\underline{x}) = \int_{\underline{x}}^{\tilde{x}} \dots \int_{\underline{x}}^{\tilde{x}} \tilde{t}(\underline{x}, s_2, \dots, s_I) dz(\pi(\tilde{\mu}, \tilde{\mu}))(s_2) \dots dz(\pi(\tilde{\mu}, \tilde{\mu}))(s_I) \leq 0,$$

because monetary transfers to a participating agent are non-positive and that

$$\tilde{V}(\underline{x}) = \sum_{k \in \mathcal{K}} b^k \tilde{Q}^k(\underline{x}) \geq 0$$

by $b^k \geq 0$ and $\tilde{Q}^k(\underline{x}) \geq 0$ for all k . Applying (52) as the underlying probability distribution over the message space in the direct mechanism to Theorem 1 in

Gershkov, et al. (2013), monetary transfers in $\mu = \{q^1, \dots, q^K, t\}$ are determined by

$$t(x, \mathbf{x}_{-1}) = \frac{\rho(\underline{x}, \mathbf{x}_{-1})}{\tilde{V}(\underline{x})} \tilde{T}(\underline{x}) + \rho(\underline{x}, \mathbf{x}_{-1})\underline{x} - \rho(x, \mathbf{x}_{-1})x + \int_{\underline{x}}^x \rho(s, \mathbf{x}_{-1})ds,$$

where ρ is defined as

$$\rho(s, \mathbf{x}_{-1}) := \sum_{k \in \mathcal{K}} b^k q^k(s, \mathbf{x}_{-1}) \geq 0$$

for all $s \in X = [\underline{x}, \bar{x}]$. Because μ is DIC, $\rho(s, \mathbf{x}_{-1})$ is non-decreasing in s .

First of all, we have

$$t(\underline{x}, \mathbf{x}_{-1}) = \frac{\rho(\underline{x}, \mathbf{x}_{-1})}{\tilde{V}(\underline{x})} \tilde{T}(\underline{x}) \leq 0. \quad (58)$$

because $\tilde{T}(\underline{x}) \leq 0$, $\tilde{V}(\underline{x}) \geq 0$ and $\rho(\underline{x}, \mathbf{x}_{-1}) \geq 0$.²³ Secondly, consider $t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1})$ for any $x, x' \in X$ with $x > x'$:

$$t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1}) = -\rho(x, \mathbf{x}_{-1})x + \rho(x', \mathbf{x}_{-1})x' + \int_{x'}^x \rho(s, \mathbf{x}_{-1})ds \leq 0. \quad (59)$$

The inequality holds because we have

$$\rho(x, \mathbf{x})x - \rho(x', \mathbf{x})x' \geq \int_{x'}^x \rho(s, \mathbf{x})ds$$

given that $\rho(s, \mathbf{x}_{-1})$ is non-decreasing in s with $\rho(s, \mathbf{x}_{-1}) \geq 0$ for all $s \in X \subset \mathbb{R}_+$. (58) and (59) imply that $t(x, \mathbf{x}_{-1}) \leq 0$ for all $x \in X$ given any $\mathbf{x}_{-1}$.

Finally we can reach the following conclusion. Because $(\tilde{\mu}, \tilde{\pi})$ is an allocation where it is a continuation equilibrium that agents report their true type to a principal who they select according to $\tilde{\pi}$, (μ, π) is an allocation such that (i) it is a continuation equilibrium that agents report their true type to a principal who they select according to π and (ii) each agent's interim expected payoffs and the ex-ante expected payoff for the principal with the original direct mechanism $\tilde{\mu}$ are preserved.

Appendix B: Proof of Theorem 5

First, we prove that $\bar{\phi}_J$ is that which is specified in (32). $\bar{\phi}_J$ is derived by the seller's joint profit maximization as in (10). Since all sellers post an identical

²³As in Gershkov, et al (2013), 0/0 is interpreted as 1.

direct mechanism $\mu = \{q, p\}$ in the joint profit maximization as their selling mechanism and that each buyer selects a seller according to $\pi(\mu, \mu)(x) = 1/J$, we can then derive a reduced-form direct mechanism $Q : X \rightarrow [0, 1]$ and $P : X \rightarrow \mathbb{R}_+$ similar to (4) and (5), based on $z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ according to (9). For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ .

Then, the seller's problem is to find $Q : X \rightarrow [0, 1]$ and $P : X \rightarrow \mathbb{R}_+$ that maximize (34) subject to

$$\begin{aligned} \text{(IC)} \quad & xQ(x) - P(x) \geq xQ(x') - P(x') \text{ for all } (x, x') \in [0, 1], \\ \text{(IR)} \quad & xQ(x) - P(x) \geq 0 \text{ for all } x \in [0, 1]. \end{aligned}$$

Let the interim expected utility for a buyer with valuation $x_i \in [0, 1]$ be denoted by $U(x_i) = x_iQ(x_i) - P(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payment:

$$\int_0^1 \cdots \int_0^1 x_i q(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \int_0^1 U(x_i) dz(\pi)(x_i) \quad (60)$$

By using the envelope theorem, we can show that

$$U(x_i) = U(0) + \int_0^{x_i} Q(s) ds. \quad (61)$$

Because the seller does not want to leave any unnecessary rents to buyers, we have $U(0) = 0$ at the optimum. Substituting (61) into (60) and integrating by parts yields

$$\begin{aligned} & N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \\ &= N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N), \end{aligned} \quad (62)$$

The last equality holds because

$$\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.$$

Since the incentive compatibility for buyer i is equivalent to (68), and by the monotonicity of $Q(x_i)$, the optimal mechanism maximizes (62) subject to $q(x_i, \mathbf{x}_{-i}) \geq 0$ and $\sum_{i=1}^N q(x_i, \mathbf{x}_{-i}) \leq 1$ for all $(x_i, \mathbf{x}_{-i}) \in X^N$, and $Q(\cdot)$ is non-decreasing.

Because there is no capacity limit, (62) is maximized if the seller sells its products to all buyers whose valuation is no less than x^* , that is, for all $\mathbf{x}_{-i}$

$$q(x_i, \mathbf{x}_{-i}) := Q(x_i) = \begin{cases} 1 & \text{if } x_i \geq x^*, \\ 0 & \text{otherwise.} \end{cases} \quad (63)$$

Therefore, (62) at the solution for the joint profit maximization is

$$\bar{\phi}_J = \frac{N}{J} \int_{x^*}^1 \left[\left(x - \frac{1 - F(x)}{f(x)} \right) \right] f(x) dx$$

Given (63), a buyer's payment in the optimal selling mechanism becomes (35).

The optimal selling mechanism $\mu = \{Q, P\}$, characterized by (63) and (35), is DIC because the probability and payment only depends on the buyer's own type message but not other buyers'. It is also deceptively simple to be implemented: A seller simply posts a single price x^* . Any buyer who wants the product purchases it at price x^* .

To complete the proof, let us show how to support a seller's ex-ante expected profit $\phi \in \Phi_J^* = [0, \bar{\phi}_J]$ in an equilibrium. When each seller sells his product at price x and buyers choose each seller with equal probability, a seller's ex-ante expected profit is

$$\phi = \frac{N}{J} x(1 - F(x)). \quad (64)$$

If $x = 0$ or 1 , then $\phi = 0$. If $x = x^*$, then $\phi = \bar{\phi}_J$. By the continuity of (64), then for any $\phi \in \Phi_J^* = [0, \bar{\phi}_J]$, there exists at least one $x(\phi)$ that satisfies (64)

Fix any $\phi \in \Phi_J^*$ as a seller's ex-ante expected profit. Let every seller post a deviation-reporting contract g^* specified in (33) with $H = \{0, 1\}$. If no seller deviates from g^* , all buyers report $h = 0$ to every seller whose price is $x(\phi)$. Then, buyers with valuation $x(\phi)$ or higher chooses some seller with equal probability and buys the product at price $x(\phi)$. This continuation equilibrium behavior yields the ex-ante expected payoff of ϕ for every seller.

Suppose that a seller deviates from g^* . Then, all buyers report $h = 1$ to every non-deviating seller whose price goes down to zero. A deviating seller must offer an (expected) price that is no higher than zero in order to attract any buyers. This is clearly not profitable because $\phi \geq 0$. Therefore, there is no profitable deviation to any arbitrary mechanism.

Appendix C: Proof of Theorem 8

$\bar{\phi}_J$ is reached when sellers jointly maximize their ex-ante expected profits. Let a direct mechanism μ be characterized by $\{q_i, p_i\}_{i=1}^N$, where $q_i : X^N \rightarrow [0, 1]$ and

$p_i : X^N \rightarrow \mathbb{R}_+$ specify the probability of acquiring the object and the payment to the seller, respectively. Specifically, for all $\mathbf{x} = [x_1, \dots, x_N]$ and all $i = 1, \dots, N$, $q_i(\mathbf{x})$ and $p_i(\mathbf{x})$ are the probability that buyer i acquires the object and buyer i 's payment to the seller with $q_i(0, \mathbf{x}_{-i}) = p_i(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$.

Because every seller offers an identical direct mechanism, each buyer selects each seller with equal probability, $1/J$. Then, the probability that a buyer's valuation is less than x_i or she selects another seller is given by

$$z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x. \quad (65)$$

Because each seller offers identical direct mechanisms, we can fix $z(\pi(\mu, \mu))(x_i)$ to (65) in the seller's joint profit maximization problem. For simplicity, we define

$$z(\pi)(x_i) = z(\pi(\mu, \mu))(x_i)$$

for all μ and all x_i . Then, we can derive the reduced-form direct mechanism $\{Q_i, P_i\}_{i=1}^N$ similar to (4) and (5) based on $z(\pi(\mu, \mu))$ specified in (65). It follows that the seller's joint profit maximization problem is to find a direct mechanism that maximizes her ex-ante expected profit:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N p_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \quad (66)$$

subject to

$$\begin{aligned} \text{(IC)} \quad & x_i Q_i(x_i) - P_i(x_i) \geq x_i Q_i(x'_i) - P_i(x'_i) \text{ for all } (x_i, x'_i) \in [0, 1], \\ \text{(IR)} \quad & x_i Q_i(x_i) - P_i(x_i) \geq 0 \text{ for all } x_i \in [0, 1], \end{aligned}$$

and $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$.

Let buyer i 's interim expected utility be denoted by $U_i(x_i) = x_i Q_i(x_i) - P_i(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payments:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N x_i q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \sum_{i=1}^N \int_0^1 U_i(x_i) dz(\pi)(x_i) \quad (67)$$

By using the envelope theorem, we can show that

$$U_i(x_i) = U_i(0) + \int_0^{x_i} Q_i(s) ds. \quad (68)$$

Because the seller does not want to leave any unnecessary rents to buyers, we have $U_i(0) = 0$ at the optimum. Substituting (68) into (67) and integrating by parts yields

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \quad (69)$$

$$= \int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N), \quad (70)$$

The last equality holds because

$$\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.$$

Because the incentive compatibility for buyer i is equivalent to (68), and by the monotonicity of $Q_i(x_i)$, the optimal mechanism maximizes (70) subject to $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$, and $Q_i(\cdot)$ is non-decreasing.

Because a mechanism is anonymous ($Q_i(\cdot) = Q(\cdot)$), the optimal mechanism takes the form of an auction with reserve price: If the virtual valuation, $x_i - \frac{1 - F(x_i)}{f(x_i)}$, is non-decreasing, the object goes to the highest valuation buyer if it is sold at all. It is sold if and only if

$$\max_{i \in \{1, \dots, N\}} x_i \geq x^*, \text{ with } x^* \text{ defined in (39).}$$

Because the auction is anonymous, the seller's maximum ex-ante expected profit is

$$\begin{aligned} \bar{\phi}_J &= N \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) Q(x) z'(\pi)(x) dx \\ &= \frac{N}{J} \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) \left(1 - \frac{1 - F(x)}{J} \right)^{N-1} f(x) dx. \end{aligned}$$

As we take its limit, $\bar{\phi}_\infty = \lim_{J \rightarrow \infty} \bar{\phi}_J$ is equal to the expression in (41).

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