DEPARTMENT OF ECONOMICS

WORKING PAPER SERIES

2019-09

McMASTER UNIVERSITY

Department of Economics
Kenneth Taylor Hall 426
1280 Main Street West
Hamilton, Ontario, Canada
L8S 4M4

http://www.mcmaster.ca/economics/

General Competing Mechanisms with Frictions*

Seungjin Han†
McMaster University

September 9, 2019

Abstract

This paper studies the class of robust equilibria in a general competing mechanism game for decentralized markets with frictions in which non-deviating sellers punish a deviator with dominant strategy incentive compatible (DIC) direct mechanisms. Given one-dimensional, independent, and private types, the lower bound of a seller's payoff in such equilibria is his minmax value over all DIC direct mechanisms if a seller can deviate to a contract that determines a menu of any complex mechanisms conditional on buyers' messages and he chooses a mechanism he wants from it. In applications, the number of sellers is endogenized given a number of buyers and fixed entry costs. As the number of buyer increases, a unique equilibrium emerges and the equilibrium ratio of buyers to sellers converges to the point where a seller's net profit is zero with the monopoly terms of trade. (JEL C72, D47, D82)

1 Introduction

Buyers are informed about terms of trade or prices offered by sellers in the market as they search for a better deal. Not only do they have private information on their payoff type, this implies that they also have market information. It is important for a seller to gather market information from buyers to determine his terms of trade.

It is not hard to gather market information these days due to the rapid advance in technology. This is an especially prominent feature of a seller's web design in on-line markets. On-line sellers can keep track of buyers' search history based on html cookies and most of them are based on simple binary messages: Whether or not a buyer revisits a seller's web site, whether or not a buyer clicks a certain part of a seller's web site, etc. This type of information can reveal what buyers know about

*This paper is a substantially revised version of the paper, "Neutralized Competition."

†Acknowledgement to be added. Address: Department of Economics, McMaster University, 1280 Main Street West, Hamilton, ON, CANADA. Email: hansj@mcmaster.ca

competing sellers' terms of trade. For example, the more buyers revisit a seller's web site, the more likely it reflects their intensified search for a lower price somewhere else (Peters (2015)). The number of web sites that a buyer has visited can also reflect how well a buyer is informed about the products in the market (Board and Lu (2018)).

Despite the prevalent use of buyers' search behavior or their market information in practice, it is extremely difficult to develop a tractable competition theory that reflects it. The message space in a seller's mechanism or contract must be enlarged so that it allows buyers to reveal not only their payoff type but also their market information (Epstein and Peters (1999)). Sellers can then maintain many collusive outcomes if they commit to mechanisms or contracts that allow them to punish a deviating seller when a competing seller's deviation becomes evident from buyers' messages (Yamashita (2010), Peters and Troncoso-Valverde (2013)). The lower bound of a seller's equilibrium payoff is however characterized with respect to only complex mechanisms allowed in the game but not incentive compatible direct mechanisms.

Furthermore, the standard competing mechanisms (e.g., Epstein and Peters (1999), Yamashita (2010), Xiong (2013) among many) assume that a profile of messages sent by buyers completely determines a seller's action (i.e., terms of trade or his allocation) given his contract or mechanism. Szentes (2010) shows that this is a restrictive feature of standard competing mechanism games in the case of complete information. If a seller is restricted to offer a mechanism or contract that completely determines his action conditional on buyers' messages as in standard competing mechanism games, the greatest lower bound of a seller's equilibrium payoff is his *maxmin* value over actions. This is because non-deviating sellers can choose their actions that minimize the deviator's payoff, conditional on the deviator's action reported by buyers. If a seller can offer a contract that assigns only a set of actions conditional on buyers' messages, the greatest lower bound rises to the minmax value over the whole set of actions. The reason is that a deviating seller can offer a contract that leaves the whole set of actions with him regardless of buyers' messages and subsequently he can choose an action from the whole set that maximizes his payoff given his belief on non-deviating sellers' action choices. Therefore, a seller's equilibrium payoff cannot go below his *minmax* value over all actions.

This paper introduces a *general competing mechanism game* of incomplete information for canonical models considered in competing mechanisms with frictions (Epstein and Peters (1999), McAfee (1993)): Buyers' payoff types are their private information but they are ex-ante identical, sellers are also identical, and each buyer has to choose only one seller. These canonical models endogenize frictions in the market by buyers' symmetric selection behavior in the sense that a buyer chooses the same mechanisms with equal probability. Therefore, sellers with same mechanisms may end up with havng the different numbers of buyers. This reflects the lack of coordination in a decentralized market. This paper also considers buyers' symmetric selection behavior to endogenize frictions.

In the general competing mechanism game, every seller can offer a contract that assigns a set of mechanisms conditional on buyers' messages, to let him then choose a mechanism from the chosen set. Given an array of mechanisms chosen by sellers, each buyer sends a message once again only to a seller who she eventually selects. The messages in the first round are similar to information that buyers leave (through html cookies) on sellers' web sites they visit as they search for a better deal. In this round, buyers do not need to select a seller. Sellers' web sites are then responsive in the sense that they can choose a selling scheme (i.e., mechanism such as an auction with reserve price), considering what are left in html cookies. In the second round, after observing sellers' selling schemes, each buyer sends a message only to a seller who she selects: For example, buyers submit their bids only to a seller's auction site they select after search.

Given a profile of mechanisms chosen by sellers and buyers' strategies of selecting a seller, buyer's communication strategies induce a Bayesian incentive compatible (BIC) direct mechanism from a seller's mechanism. It is very hard to work with BIC direct mechanisms because the Bayesian incentive compatibility is endogenous in that it is based on buyers' interim payoffs, which depends on buyers' selection strategies in a continuation equilibrium. This implies that Bayesian incentive compatibility of a seller's direct mechanism depends on the mechanisms that all sellers choose. On the other hand, it is easy to work with the dominant strategy incentive compatible (DIC) direct mechanisms because it can be defined without reference to buyers' selection strategies or other sellers' mechanisms. For that reason, we focus on a class of equilibria where non-deviating sellers punish a deviating seller with DIC direct mechanisms. We do not provide the full characterization of all equilibria but the class of equilibria we focus here can be used in various applications considered for decentralized markets with frictions.

One may ask if it is also okay to restrict a deviating seller's deviation to DIC direct mechanisms. If buyers' types are one-dimensional, private and independent and payoff functions are linear, we can extend the BIC-DIC equivalence established by Gershkov, et. al. (2013) to show that any BIC direct mechanism induced by a deviating seller's mechanism and buyers' communication strategies can be replaced by an (interim) payoff-preserving DIC direct mechanism in a given continuation equilibrium. For the class of equilibria where non-deviators punishes a deviating seller with DIC direct mechanisms, we utilize this BIC-DIC equivalence in establishing that every seller can restrict himself to offer a contract that specifies a set of DIC direct mechanisms conditional on buyers' messages on and off the path, even if a seller can offer a contract that assigns a set of any arbitrary mechanisms, not just DIC direct mechanisms.[1]

As in Epstein and Peters (1999), McAfee (1993), and most studies in competing auctions, this paper focuses on symmetric solutions of the model with respect to allocations, equilibria, etc. For the characterization of the lower bound of a seller's equilibrium payoff, note that a deviating principal can deviate to a contract that

[1]In the case of complete information, the general competing mechanism game considered in our paper is the same as one considered in Szentes (2010).

assigns the whole set of his DIC direct mechanisms regardless of buyers' messages. He can then choose a DIC direct mechanism that maximizes his payoff given his belief on DIC direct mechanisms chosen by non-deviating sellers. This makes the lower bound of a seller's equilibrium payoff equal to his *minmax* value, instead of maxmin, over all DIC direct mechanisms.[2] Given this lower bound, the set of a seller's equilibrium payoffs in our general competing mechanism game is shown to be a connected interval between the lower bound and the upper bound, which is a seller's expected profit achieve by the joint profit maximization.

This paper considers an application where sellers can produce each unit at a constant marginal cost and each buyer has a unit demand with private valuation. If sellers have no capacity constraint, they can lower their price down to their marginal cost upon a competing seller's deviation reported in their deviation-reporting contracts. This implies that any seller must incur loss in order to attract buyers upon deviation to any arbitrary mechanism. Therefore, the lower bound of a seller's equilibrium profit can be as low as a seller's reservation profit. If the hazard rate of the buyer's valuation is non-decreasing, the upper bound of a seller's equilibrium profit is reached at the monopoly price, i.e., the buyer's valuation that makes her virtual valuation equal to zero. This analysis is based on a fixed number of buyers and sellers. However, sellers may be free to enter the market by incurring a fixed cost. Given a number of buyers, the equilibrium number of sellers and therefore the equilibrium ratio of buyers to sellers are determined at the point where one additional seller in the market yields a negative profit net of a fixed cost even in the joint profit maximization.

Given a finite number of buyers, the equilibrium number of sellers may generate a non-degenerate range of equilibrium profits and the corresponding range of prices. As the number of buyers goes to infinity, the equilibrium ratio of buyers to sellers yields the monopoly price as the unique equilibrium price but their equilibrium profit is zero. This is quite a surprising result, which contrasts to the result from the standard game theoretical analysis or neoclassical microeconomic theory. The intuition for our result is as follows. In equilibrium, each seller serves the equal proportion of the market in expectation, which is the ratio of the number of buyers to the number of sellers. There is a unique ratio that makes each seller's expected gross profit equal to his fixed entry cost in joint profit maximization. With a finite number of buyers, the equilibrium ratio of buyers to sellers may not the same as that unique ratio. As the number of buyer increases, the equilibrium ratio of buyers to sellers converges to it, resulting in the monopoly price but leaving zero net profits with every seller.

This large market result holds even when each seller faces a capacity constraint. In a canonical environment for competing auctions (Burguet and Sákovics (1999), Han (2015), McAfee (1993), Peters and Severinov (1997), Peters (1997), Virag (2010)), the joint profit maximization occurs when each seller offers the monopoly auction, i.e., one with reserve price equal to the buyer's valuation at which her virtual valuation is zero given the monotone hazard rate. We consider a large market where the number of

[2]The set of all DIC direct mechanisms available for each seller is independently defined.

buyers is taken to infinity with a fixed ratio of buyers to sellers. If we endogenize the ratio of buyers and sellers, it is uniquely determined at the point where each seller's expected profit net of a fixed entry cost is zero and the equilibrium mechanism is unique in the sense that it is the monopoly auction.

2 Preliminaries

If necessary, we assume that a set X represents a compact metric space. Where only a weaker structure is needed, it will be made explicit. When a measurable structure is necessary, the corresponding Borel σ - algebra is used.

J principals ($J \geq 2$, e.g., sellers) compete in the market with N agent (e.g., buyers) with $N \geq 3$. Throughout the paper, we use the terms, sellers and buyers instead of principals and agents for ease of exposition but the model can actually capture other principal/agent applications. Each buyer's type is independently drawn from a probability distribution F with support $X = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$ and it is not observed by anyone else.[3] Each seller makes an allocation decision for buyers who select him. We assume that each seller's allocation decision is to choose (i) one of his action alternatives from the finite set, $\mathcal{K} = \{1, \ldots, K\}$, and (ii) monetary transfers to buyers who select him.

All buyers are ex-ante identical. An buyer's payoff depends on the action taken by the seller she selects and monetary transfer. The payoff for a buyer of type x associated with choosing a seller who takes action alternative k is

$$b^k x + g^k + t, \tag{1}$$

where $b^k \geq 0$ and $g^k \in \mathbb{R}$ for all $k \in \mathcal{K}$ and $t \leq 0$ is a monetary transfer to the buyer.[4] For example, k is the identity of the winning bidder in an auction environment. In this case, bidder n has the following preference parameter values: $b^k = 1$ if $k = n$, $b^k = 0$ otherwise; $g^k = 0$ for all k. If a buyer does not choose any seller, she receives her reservation payoff, which is equal to zero.

A seller's payoff depends on his choice of action alternative k and monetary transfer ℓ;

$$a^k w + y^k + \ell, \tag{2}$$

where $a^k, y^k \in \mathbb{R}$ for all $k \in \mathcal{K}$, $w \in \mathbb{R}$. We assume that $\max_k [a^k w + y^k] \geq 0$. w can be seen as a seller's type, which is publicly known and the same to every seller. The homogeneity of sellers and buyers is for simplicity but it can be relaxed.

[3] We use masculine pronouns for sellers and feminine pronouns for buyers.

[4] We consider the cases where a buyer (agent) pays a positive amount of money to a seller (principal). Because t is a monetary transfer to a buyer (agent), it means that t is assumed to be non-positive. The assumption of $t \leq 0$ essentially implies that a seller (principal) cannot make a positive amount of monetary transfer to a buyer. This is what we usually observe in practice. We can assume $t \geq 0$ for the cases where buyers are principals and sellers are agents.

For an auction environment, we can set $\mathcal{K} = \{1, \ldots, N, N+1\}$ so that, given N potential bidders, alternative k implies that the winning bidder is bidder k if $k \le N$, but $k = N+1$ means that the seller retains the object. The seller has then the following preference parameter values: $a^k = 0$ for $k \le N$ and $a^k = 1$ for $k = N+1$; $y^k = 0$ for all k. w represents the value of the object to the seller. A seller's reservation payoff is equal to zero. We impose the *budget balance* condition for each seller so that the sum of ℓ and monetary transfers t to buyers who choose him is equal to zero. This implies that $\ell \ge 0$. Non-positive monetary transfer to a buyer and non-negative monetary transfer to a seller reflect limited liability on both buyers and sellers.

3 Feasible Allocations

An (interim) allocation should tell us how each seller's action and monetary transfers depend on the participating buyers' types and how each buyer selects a sellers. In that sense, an allocation is characterized by (a) an array of direct mechanisms offered by sellers and (b) the buyer's selection behavior that decides how to select a seller among J sellers given an array of direct mechanisms.

If a buyer does not select a seller, he treats her type as x°. Let $\bar{X} = X \cup \{x^\circ\}$. For any given seller, $\mathbf{x} \in \bar{X}^N$ can conveniently characterize the type profile of the buyers who select him. Let a seller's direct mechanism μ be denoted by $\{q^1, \ldots, q^K, t\}$, where $q^k : \bar{X}^N \to [0,1]$ determines the probability of alternative k as a function of buyers' type messages and $t : \bar{X}^N \to \mathbb{R}_-$ determines an amount of monetary transfer to a buyer as a function of buyers' type messages. We assume that mechanisms are *anonymous* with respect to buyers so that they cannot distinguish among different buyers except on a basis of messages sent by buyers to the mechanisms. Sellers are also not distinguished except on the basis of their mechanisms (and contracts). We are interested in a (*symmetric*) *allocation* where (i) buyers choose sellers with equal probability as long as their mechanisms are the same and (ii) every seller's direct mechanism is identical.

We first construct a buyer's selection behavior given an array of direct mechanisms when one seller offers μ' and all the other sellers offer μ. Let $\pi(\mu', \mu)(x) \in [0,1]$ denote the probability with which each buyer of type x selects the seller who offers μ' when all the other sellers offer μ. Define $z(\pi(\mu', \mu))(x)$ as follows

$$z(\pi(\mu', \mu))(x) := 1 - \int_x^{\bar{x}} \pi(\mu', \mu)(s) dF. \tag{3}$$

The term $z(\pi(\mu', \mu))(x)$ is the probability that a buyer either has her type below x as a participant of μ' or selects any other seller whose mechanism is μ.

An array of BIC (Bayesian incentive compatible) direct mechanisms (μ', μ) chosen by sellers defines a subgame that buyers play.[5] A (truthful symmetric) *continuation*

[5] (μ', μ) means that one principal offers μ' and all the other principals offer μ.

equilibrium at a subgame (μ', μ) can be characterized by (a) every buyer's selection strategy $\pi(\mu', \mu)$ and (b) Bayesian incentive compatibility of direct mechanisms.[6] Let $\Pi(\mu', \mu)$ be the set of all possible continuation equilibria (i.e., the set of all optimal selection strategies that buyers choose in all continuation equilibria at a subgame (μ', μ)). Given $\mu' = \{\acute{q}^1, \ldots, \acute{q}^K, \acute{t}\}$, we can derive the reduced-form direct mechanism $\{\acute{Q}^1, \ldots, \acute{Q}^K, \acute{T}\}$ such that, for all $x \in X$,

$$\acute{Q}^k(x) := \int_{\underline{x}}^{\bar{x}} \cdots \int_{\underline{x}}^{\bar{x}} \acute{q}^k(x, s_2, \ldots, s_I) dz(\pi(\mu', \mu))(s_2) \ldots dz(\pi(\mu', \mu))(s_I), \tag{4}$$

$$\acute{T}(x) := \int_{\underline{x}}^{\bar{x}} \cdots \int_{\underline{x}}^{\bar{x}} \acute{t}(x, s_2, \ldots, s_I) dz(\pi(\mu', \mu))(s_2) \ldots dz(\pi(\mu', \mu))(s_I). \tag{5}$$

The interim expected payoff for the buyer of type x associated with reporting the true type upon selecting the seller whose direct mechanism is μ' is

$$U_J(\mu', \mu, \pi, x) := \sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x) + \acute{T}(x). \tag{6}$$

μ' is Bayesian incentive compatible (BIC) given $\pi(\mu', \mu)$ if for all $x, x' \in X$

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x) + \acute{T}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x') + \acute{T}(x') \tag{7}$$

As shown in (7) above, incentive compatibility is imposed only for all $x, x' \in X$ but not $\bar{X}$, i.e., truth telling is optimal only for those buyers participating in the seller's mechanism. If a buyer does not select the seller, we treat her effective type as x° in $\bar{X}$ and there is no type reporting for her. The ex-ante expected payoff for the seller whose direct mechanism is μ' is[7]

$$\Phi_J(\mu', \mu, \pi) := \\ \sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \acute{Q}^k(s_1) dz(\pi(\mu', \mu))(s_1) - N \int_{\underline{x}}^{\bar{x}} \acute{T}(s_1) dz(\pi(\mu', \mu))(s_1). \tag{8}$$

We define a (symmetric) feasible allocation as follows.

Definition 1 (μ, π) *is a (symmetric) feasible allocation if (i)* μ *is BIC given* $\pi(\mu, \mu)$*, (ii) a buyer chooses each seller according to* $\pi(\mu, \mu)(x) = \frac{1}{J}$ *whenever* $\pi(\mu, \mu)(x) > 0$*. (iii) for all* $x \in X$*,* $\pi(\mu, \mu)(x) > 0$ *if* $U_J(\mu, \mu, \pi, x) \geq 0$*, and (iv)* $\Phi_J(\mu, \mu, \pi) \geq 0$*.*

[6] In a (symmetric) continuation equilibrium at a subgame (μ', μ), an agent of type x selects a principal whose mechanism is μ with probability $\frac{1-\pi(\mu', \mu)(x)}{J-1}$ if she selects a principal with μ.

[7] The subscript in $\Phi_J(\mu', \mu, \pi)$ and $U_J(\mu', \mu, \pi, x)$ denotes the number of principals in the market.

In a feasible allocation, conditions (i) and (ii) imply that buyers must play a continuation equilibrium in the subgame which all sellers' mechanisms are identical. Conditions (iii) and (iv) imply that a feasible allocation must be *individually rational* for both buyers and sellers. Let Z be the set of all feasible allocations.

We are first interested in the set of a seller's ex-ante expected payoffs associated with all possible feasible allocations in Z. It cannot be lower than a seller's reservation payoff, which is zero. Therefore, the minimum of a seller's ex-ante expected payoff is zero. Given symmetry in allocation, the maximum of a seller's ex-ante expected payoff can be derived by solving the joint payoff maximization problem.

There may be a buyer who decides not to choose any seller given a direct mechanism μ chosen by all sellers. If that's the case, we assume that a buyer first selects one of the sellers with equal probability $1/J$ and sends the type message x°. Therefore, in the joint payoff maximization problem, we fix $\pi(\mu, \mu)(x) = \frac{1}{J}$ for all $x \in X$. Then,

$$z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x \in X. \tag{9}$$

Given a direct mechanism $\mu = \{q^1, \ldots, q^K, t\}$, let $\{Q^1, \ldots, Q^K, T\}$ be the reduced-form direct mechanism that is derived according to (4) and (5) with (9). Then, the joint payoff maximization problem is

$$\max_{\mu} \Phi_J(\mu, \mu, \pi) \tag{10}$$

subject to

$$\begin{aligned}
\text{(IC)} \ \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x) + T(x) \ &\geq \ \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x') + T(x') \ \forall x, x' \in X, \\
\text{(IR)} \ \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k(x) + T(x) \ &\geq \ 0 \ \forall x \in X,
\end{aligned}$$

and $q^k(\mathbf{x}) \geq 0$ and $\sum_{k=1}^{K} q^k(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in \bar{X}^N$. We assume that a solution, denoted by $\bar{\mu} = \{\bar{q}^1, \ldots, \bar{q}^K, \bar{t}\}$, to the joint payoff maximization problem exists and let $\bar{\phi}_J := \Phi_J(\bar{\mu}, \bar{\mu}, \pi)$. Let Φ_J^* be the set of the seller's feasible ex-ante expected payoffs:

$$\Phi_J^* := \{\phi \in \mathbb{R} : \phi = \Phi_J(\mu, \mu, \pi) \ \forall \, (\mu, \pi) \in Z\}.$$

Lemma 1 $\Phi_J^* = [0, \bar{\phi}_J]$.

Proof. It is clear that a seller's ex-ante expected payoff associated with a feasible allocation cannot be less than zero and greater than $\bar{\phi}_J$. We only need to show that Φ_J^* is the connected interval between the two. Note that zero payoff can be achieved by

$$\mu_\circ := \{q_\circ^1, \ldots, q_\circ^K, t_\circ\} \text{ with } q_\circ^k(\mathbf{x}) = t_\circ(\mathbf{x}) = 0 \ \forall (k, \mathbf{x}) \in \mathcal{K} \times \bar{X}^N. \tag{11}$$

For any $\phi \in [0, \bar{\phi}_J]$, there exists a scalar $\alpha \in [0,1]$ such that $\phi = \alpha\bar{\phi}_J$. Given α, construct a direct mechanism

$$\mu = (1-\alpha)\mu_\circ + \alpha\bar{\mu}. \tag{12}$$

Then, $(\mu, \pi) \in Z$ with π based on (9). This is because any convex combination between two BIC direct mechanisms given the same type distribution is also BIC. Accordingly, a seller's ex-ante expected payoff is

$$\Phi_J(\mu, \mu, \pi) = \alpha\Phi_J(\bar{\mu}, \bar{\mu}, \pi) = \alpha\bar{\phi}_J.$$

Therefore, the set of the seller's feasible ex-ante expected payoffs is the closed connected interval between 0 and $\bar{\phi}_J$. ■

The Bayesian incentive compatibility of a seller's direct mechanism is based on the buyer's interim expected payoff, which depends on the probability that buyers select the seller. Because this selection probability depends on the other sellers' mechanisms as well, Bayesian incentive compatibility generally depends on the other sellers' mechanisms, which makes it difficult to work with BIC direct mechanisms. On the other hand, a dominant strategy incentive compatible (DIC) direct mechanism can be easily used because the dominant strategy incentive compatibility does not depend on buyers' selection strategies. A direct mechanism $\mu = \{q^1, \ldots, q^K, t\}$ is DIC if for all $x, x' \in X$ and all $\mathbf{x}_{-1} = (x_2, \ldots, x_I) \in \bar{X}^{N-1}$

$$\sum_{k\in\mathcal{K}} \left(b^k x + g^k\right) q^k(x, \mathbf{x}_{-1}) + t(x, \mathbf{x}_{-1}) \geq \sum_{k\in\mathcal{K}} \left(b^k x + g^k\right) q^k(x', \mathbf{x}_{-1}) + t(x', \mathbf{x}_{-1}).$$

As shown above, dominant strategy incentive compatibility is based on the buyer's ex-post payoff after buyers select the seller and hence a DIC direct mechanism can be defined without reference to other sellers' mechanisms. Let Ω_D be the set of all DIC direct mechanisms. By using the BIC-DIC equivalence in Gershkov, et. al. (2013), we can show that any payoff in Φ_J^* can be supported by a DIC allocation.

Corollary 1 *For any $(\tilde{\mu}, \tilde{\pi}) \in Z$, there exists a DIC allocation $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$ such that*

$$\begin{aligned} U_J(\mu, \mu, \pi, x) &= U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x) \text{ for all } x, \\ \Phi_J(\mu, \mu, \pi) &= \Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}). \end{aligned}$$

Therefore, any ϕ in Φ_J^* can be supported by a DIC allocation, i.e., $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$.

Proof. See Appendix A. ■

Corollary 1 shows that we can focus on a DIC allocation for any feasible ex-ante expected payoff in Φ_J^*. It is also worth noting that Corollary 1 holds whether or not buyers and sellers have limited liability. When they have limited liability, as specified in Section 2, we need to show that $\tilde{\mu}$ also satisfies limited liability given μ that satisfies limited liability.

4 General Competing Mechanisms

Given a profile of mechanisms chosen by sellers and buyers' strategies for selecting a seller, buyer's communication strategies generally induce a BIC direct mechanism from a seller's mechanism. It is very hard to work with BIC direct mechanisms because the Bayesian incentive compatibility is endogenous in that it is based on buyers' interim payoffs, which depends on buyers' selection strategies in a continuation equilibrium. This implies that the Bayesian incentive compatibility of a non-deviator's direct mechanism depends on the deviating seller's mechanism, so the deviator cannot take non-deviators' BIC direct mechanisms as given. On the other hand, it is easy to work with the dominant strategy incentive compatible (DIC) direct mechanisms because it can be defined without reference to buyers' selection strategies or contracts offered by other sellers. For that reason, we are not interested in all possible equilibria but we rather focus on a class of equilibria where non-deviating sellers punish a deviating seller with DIC direct mechanisms.

Generally, buyers' communication strategies given a deviating seller's mechanism induces a BIC direct mechanism. Given one-dimensional, private and independent types, Section 5 uses Corollary 1 to show that one can also focus on DIC direct mechanisms for a seller's deviation.[8] Therefore, we consider a competing mechanism game where a seller offers a contract that specifies a set (menu) of DIC direct mechanisms conditional on buyers' messages and he chooses a DIC direct mechanism he wants from the menu.

Let $\mathcal{P}_D$ be the set of all (closed) subsets of Ω_D. One can think of a typical element in $\mathcal{P}_D$ as a menu of DIC direct mechanisms. Following the terminology in Szentes (2010), we formulate a seller's (anonymous) *contract*, which is a mapping $g : H^N \to \mathcal{P}_D$, where H is a set of messages available for each buyer. Let G_D be the set of all possible contracts available for each seller. A general competing mechanism game relative to G_D unfolds in five stages as follows[9]:

S1. Principals simultaneously post their contracts.

S2. Given an array of contracts, buyers simultaneously send their messages in H to all sellers.

S3. Each seller simultaneously chooses a DIC direct mechanism from the menu that is determined by an array of messages sent by buyers given his contract.

[8]In particular, Section 5 shows that the set of PBE allocations in a general competing mechanism game relative to G_D is the set of *robust* PBE allocations supportable with DIC punishment in a general competing mechanism game relative to G_Γ, where a seller can offer a contract that specifies a set of arbitrary mechanisms conditional on buyers' messages.

[9]In the case of complete information, the general competing mechanism game described above is the same as the one considered in Szentes (2010), where a seller's contract determines a set of actions conditional on buyers' messages and the seller choose an action from the set. The reason is that the set of all DIC direct mechanisms is the set of all actions in the case of complete information where each buyer's type is drawn from a degenerate probability distribution.

S4. After observing an array of DIC direct mechanisms, each buyer selects only one seller, if any, and sends her true type to the seller she selects.

S5. Allocation decisions are determined by DIC direct mechanisms according to type messages and payoffs are realized.

The general competing mechanism game above may not exactly describe what is happening in on-line markets, but Stage 2 is similar to the search stage where buyers leave various information through html cookies on sellers' web sites they visit as they search. In Stage 2, they do not need to select a seller. Stage 3 shows that sellers' web sites are responsive in the sense that they can choose a selling scheme (i.e., mechanism), considering what are left in html cookies. Stage 4 can be thought of as the messages that buyers send to the seller whom they select: They are similar to, for example, buyers' bids submitted to a seller's auction site they select after search.

Our solution concept for an equilibrium of a general competing mechanism game relative to G_D is a symmetric pure-strategy perfect Bayesian equilibrium (henceforth simply equilibrium unless specified) in which (i) sellers use the same equilibrium pure strategies and (ii) buyers play the (symmetric) continuation equilibrium of selecting a seller that is best for a deviating seller upon his deviation. From now on, we simply call it an equilibrium unless specified. The first requirement implies that sellers use the same equilibrium strategy on the equilibrium path and that non-deviating sellers use the same strategy to punish a deviating seller. The second requirement means that there is no continuation equilibrium of selecting a seller where a seller gains upon his deviation.

We are interested in the set of a seller's ex-ante expected payoffs that can be supported in an equilibrium of a general competing mechanism game relative to G_D. We first construct a selection behavior in stage 4 when one seller's DIC direct mechanism is μ' and the other sellers all use a DIC direct mechanism μ.Let us define $\underline{\phi}_J$ as follows:

$$\underline{\phi}_J := \inf_{\mu\in\Omega_D}\left[\sup_{\mu'\in\Omega_D}\left(\sup_{\pi(\mu',\mu)\in\Pi(\mu',\mu)}\Phi_J(\mu',\mu,\pi)\right)\right]. \tag{13}$$

Note that the supremum is taken over $\pi(\mu',\mu)\in\Pi(\mu',\mu)$. The idea is that there should not be a continuation-equilibrium strategy of selecting a seller upon a seller's deviation where he gains. This is the second requirement of our solution concept.

Note that $\underline{\phi}_J$ may take the form of the following expression:

$$\min_{\mu\in\Omega_D}\left[\max_{\mu'\in\Omega_D}\left(\max_{\pi(\mu',\mu)\in\Pi(\mu',\mu)}\Phi_J(\mu',\mu,\pi)\right)\right] \tag{14}$$

However, (14) may or may not exist. Let Φ_J^e be the set of a seller's ex-ante expected payoffs that can be supported in an equilibrium of a competing mechanism game relative to G_D

Theorem 1 *If (14) does not exist,* $\Phi^e_J = (\underline{\phi}_J, \bar{\phi}_J]$. *If it does,* $\Phi^e_J := [\underline{\phi}_J, \bar{\phi}_J]$

Proof. We first prove that a DIC allocation, (μ^*, π^*) with $\mu^* \in \Omega_D$, can be supported in an equilibrium of a general competing mechanism game relative to G_D if and only if

$$\Phi_J(\mu^*, \mu^*, \pi^*) > \underline{\phi}_J \tag{15}$$

when (14) does not exist (weak inequality for (15) when (14) exists).

Consider the case where (14) does not exist. Let us prove the "only if" part by contradiction. Suppose that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D but a seller's equilibrium ex-ante expected payoff does not satisfy (15). Because $\underline{\phi}_J$ cannot be reached, it implies that

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \underline{\phi}_J = \inf_{\mu \in \Omega_D} \left[\sup_{\mu' \in \Omega_D} \left(\sup_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \tag{16}$$

A seller can deviate to a contract $\tilde{g}$ that leaves the whole set of DIC direct mechanisms Ω_D regardless of buyers' messages in H^N. Given a DIC direct mechanism μ that each non-deviating seller chooses in a continuation equilibrium upon a seller's deviation to such a contract $\tilde{g}$, the deviator can then choose μ'' from Ω_D and $\pi''(\mu'', \mu) \in \Pi(\mu'', \mu)$ such that

$$\left| \Phi_J(\mu'', \mu, \pi'') - \underline{\phi}_J \right| < \epsilon. \tag{17}$$

Combining (16) and (17) yields

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \Phi_J(\mu'', \mu, \pi'').$$

This implies that there exists a continuation equilibrium where a seller gains upon deviation. This contradicts that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D. Therefore, (15) must be satisfied.

If (14) exists, then the expression on the right hand side of the equality in (16) is (14). Given a DIC direct mechanism μ that each non-deviating seller chooses in a continuation equilibrium upon a seller's deviation to such a contract $\tilde{g}$, the deviator can then choose μ' from Ω_D and $\pi'(\mu', \mu) \in \Pi(\mu', \mu)$ such that

$$\mu' \in \arg\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right).$$

Because such a deviation must not be profitable in equilibrium, we have that

$$\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \leq \Phi_J(\mu^*, \mu^*, \pi^*) \tag{18}$$

Since the expression on the right hand side of the equality in (16) is (14), combining (16) and (18) yields

$$\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) < \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right],$$

which cannot be true. Therefore, (15) must be satisfied with weak inequality.

Now we prove the "if" part, that is, if (μ^*, π^*) satisfies (15), then it can be supported in a pure-strategy equilibrium of a competing mechanism game relative to G. First, consider the case where (14) does not exist. Divide the message space H of g to two non-empty disjoint subsets, B and B^c such that $B \cup B^c = H$. Let h_i denote buyer i's message in H. All sellers post a contract g^* such that

$$g^*(h_1, \ldots, h_n) := \begin{cases} \mu_p \in \Omega_D & \text{if } |\{i : h_i \in B\}| > N/2, \\ \mu^* \in \Omega_D & \text{otherwise,} \end{cases} \tag{19}$$

where μ_p satisfies

$$\sup_{\mu' \in \Omega_D} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu_p, \pi) \right) < \Phi_J(\mu^*, \mu^*, \pi^*) \tag{20}$$

If *all* sellers post g^*, then all buyers send messages in B^c and g^* assigns μ^*. Agents select sellers according to $\pi^*(\mu^*, \mu^*)$ and send their true types to a seller that they select. If one seller deviates to g, then all buyers send messages in B to the non-deviating seller. The non-deviator's contract g^* then assigns μ_p. Because (μ^*, π^*) satisfies (15), the deviating seller cannot gain by choosing any DIC direct mechanism, given each non-deviator's DIC direct mechanism μ_p, satisfying (20).

Now consider the case where (14) exists. We can prove that a DIC allocation, (μ^*, π^*) with $\mu^* \in \Omega_D$, can be supported in an equilibrium of a general competing mechanism game relative to G_D if and only if (15) holds with weak inequality. For the proof of the "only if" part, suppose that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D but a seller's equilibrium ex-ante expected payoff does not satisfy (15) with weak inequality:

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \underline{\phi}_J = \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right]$$

If (14) exists, we can choose μ_p that satisfies

$$\mu_p \in \arg\min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right].$$

Then, $\underline{\phi}_J$ can be supported as a seller's equilibrium ex-ante expected payoff.

What we proved above is that any DIC allocation that generates a seller's ex-ante expected payoff no less than $\underline{\phi}_J$ can be supported in a pure-strategy equilibrium.

Clearly, $\bar{\phi}_J$ is the maximum. We need to prove that any payoff between $\underline{\phi}_J$ and $\bar{\phi}_J$ is supportable in an equilibrium of a competing mechanism game relative to G to complete the proof. Note that $\Phi^e_J = [\underline{\phi}_J, \bar{\phi}_J] \subset \Phi^*_J$ because $\underline{\phi}_J \geq 0$. According to Corollary 1, any ϕ in Φ^e_J can be induced by some DIC allocation, $(\mu^*, \pi^*) \in Z$ with $\mu^* \in \Omega_D$. Therefore, any ϕ in Φ^e_J can be supported in an equilibrium where every seller posts a contract g^* that assigns the corresponding μ^* when no seller deviates. ■

It is helpful to explain the intuition behind Theorem 1 when (14) exists. If a contract specifies a single DIC direct mechanism, instead of a menu of DIC direct mechanisms, conditional on the buyer's messages, it reduces the lower bound of a seller's equilibrium ex-ante expected payoff down to the maxmin value instead of the minmax. This is because, given the deviator's DIC direct mechanism, non-deviators choose their DIC direct mechanisms to minimize the deviator's payoff.

In a general competing mechanism game relative to G_D described above, a seller's contract specifies a menu of DIC direct mechanisms conditional on the buyers' messages, from which he chooses his own DIC direct mechanism. This makes it possible for a deviating seller to choose a DIC direct mechanism that maximizes his ex-ante expected payoff given a profile of non-deviators' DIC direct mechanisms and hence the lower bound of a seller's equilibrium ex-ante expected payoff in a general competing mechanism game relative to G_D rises to the minmax value in terms of DIC direct mechanisms.

It is worth mentioning a few points related to Theorem 1. Let us define $\Lambda_J(\mu', \mu)$ as

$$\Lambda_J(\mu', \mu) := \sup_{\pi(\mu',\mu)\in\Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi)$$

Then, the inf-sup value in (13) is expressed as $\inf_{\mu\in\Omega_D} \sup_{\mu'\in\Omega_D} \Lambda_J(\mu', \mu)$. Suppose that we consider a competing mechanism game where sellers offer standard mechanisms that specify a single DIC direct mechanism, instead of a menu of DIC direct mechanisms, conditional on buyers' messages in H^{N-1}.Then, we can show that the lower bound of a seller's equilibrium payoff is $\sup_{\mu'\in\Omega_D} \inf_{\mu\in\Omega_D} \Lambda_J(\mu', \mu)$ in such a game, whereas it is $\inf_{\mu\in\Omega_D} \sup_{\mu'\in\Omega_D} \Lambda_J(\mu', \mu)$ in the general competing mechanism game where a seller's mechanism specifies a menu of DIC direct mechanisms conditional on buyers' messages in H^N. Generally, $\sup_{\mu'\in\Omega_D} \inf_{\mu\in\Omega_D} \Lambda_J(\mu', \mu)$ is no greater than $\inf_{\mu\in\Omega_D} \sup_{\mu'\in\Omega_D} \Lambda_J(\mu', \mu)$.

If (i) Ω_D were compact and convex, and (ii) Λ_J were to be quasi-concave, upper semi-continuous in μ and quasi-convex, lower semi-continuous in μ', we could apply Sion's minimax theorem (1958) to swap the inf and sup. Then, it may be redundant to give a seller freedom to offer a contract that specifies a menu of DIC direct mechanisms conditional on buyers' messages. Unfortunately, Ω_D is not compact. It is also not known whether or not Λ_J is quasi-concave, upper semi-continuous in μ and quasi-convex, lower semi-continuous in μ'. Therefore, one cannot simply swap the inf and sup.

Second, (19) shows that it is sufficient to have a message space H which includes only two messages. Note that we are focused on symmetric equilibria and sellers are all identical. Therefore, it is enough to have two messages in H because the role of the messages is to reveal whether or not a competing seller deviates. For this, let $H = \{0, 1\}$, $B = \{1\}$, and $B^c = \{0\}$ so that $B \cup B^c = H$. The message 1 implies that the competing seller deviated. The message 0 means that the competing seller did not. Binary messages can be easily adopted in practice (e.g., on-line markets). Let us call g^* with $H = \{0, 1\}$ a deviation-reporting contract. If sellers are heterogenous, then non-deviating sellers need to know the identity of a deviating seller. In this case, H should be $\{0, 1, \ldots, J\}$ where a non-zero number represents the identity of a deviating seller and zero means no deviation by any competing seller.

Lastly, it is also worthwhile to mention the role of three or more buyers. With three or more buyers, truthful reporting on whether or not there is a deviation by a competing seller can be supported in a continuation equilibrium. Given the deviation-reporting contract in (19) and three or more buyers, any single buyer's deviation from truth telling does not change the choice of a DIC direct mechanism when everyone else truthfully choose a message in H. This is a trick Yamashita employed in his paper (2010).[10]

5 Equilibrium Allocations with DIC Punishment

A contract defined in the previous section assigns only a menu of DIC direct mechanisms contingent on messages sent by buyers. This may restrict a deviating seller's ability to come up with a better mechanism that could provide a higher ex-ante expected payoff. Let $\gamma = \{\sigma^1, \ldots, \sigma^K, \tau\}$ be a seller's arbitrary (anonymous) mechanism with a message space $\bar{M} = M \times \{m^\circ\}$ for each buyer.[11] For all $k \in \mathcal{K}$, $\sigma^k : \bar{M}^N \to [0, 1]$ specifies the probability of alternative k as the function of buyers' messages in $\bar{M}^N$. $\tau : \bar{M}^N \to \mathbb{R}_-$ specifies monetary transfer to a buyer as the function of buyers' messages in $\bar{M}^N$. Let Γ be the set of all possible mechanisms with message space $\bar{M}$. One may wonder if an equilibrium DIC allocation in a general competing mechanism game relative to G_D is also an equilibrium allocation in a general competing mechanism game relative to G_Γ where G_Γ is the set of all possible contracts that specify a menu of mechanisms in Γ conditional on buyers' messages in H^N.

Let us fix an equilibrium DIC allocation (μ^*, π^*) in a general competing mechanism game relative to G_D and let $g^* \in G_D$ that supports (μ^*, π^*). We assume that Γ is large enough that the set of all direct mechanisms Ω, including all DIC direct mechanisms, can be embedded into Γ. Given our assumption on Γ, it is also clear that $g^* \in G_\Gamma$

[10] If there is monetary transfer with no limited liability, truthful reporting on a competing seller's deviation can be supported even with two buyers. If their messages from H are inconsistent, a seller can impose a huge amount of transfers from them.

[11] Not participating is equivalent to sending m°.

with a slight abuse of notation (More precisely, there exists a contract g' in G_Γ such that g^* is homeomorphic to it).

When a seller deviates, each non-deviating seller's contract g^* will choose a DIC direct mechanism μ_p given that buyers all report that his competitor deviated in a continuation. Suppose that a deviating seller posts a contract g that assigns a menu of mechanisms in Γ and chooses a mechanism γ from the menu that is assigned. Then, an array of mechanisms (γ, μ_p) defines a subgame played by buyers.[12] We can fix truthful type reporting to each non-deviating seller because μ_p is DIC. Then, a continuation equilibrium of the subgame defined by (γ, μ_p) is characterized by the buyer's strategy of communicating with the deviating seller, $c(\gamma, \mu_p) : X \to \Delta(M)$ upon selecting him and her strategy of selecting him, $\pi(\gamma, \mu_p) : X \to [0, 1]$.

The buyer's communication strategy $c(\gamma, \mu_p)$ induces a direct mechanism $\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ from $\gamma = \{\sigma^1, \ldots, \sigma^K, \tau\}$. Let I denote the number of buyers who select the deviating seller. Then, for every $I \leq N$ and every $(x_1, \ldots, x_I) \in X^I$, $\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ is defined as, for all $k \in \mathcal{K}$,

$$q^k_{c,\mu_p}(x_1, \ldots, x_I, \mathbf{x}^\circ_{-I}) = \int_M \cdots \int_M \sigma^k(m_1, \ldots, m_I, \mathbf{m}^\circ_{-I}) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

$$t_{c,\mu_p}(x_1, \ldots, x_I, \mathbf{x}^\circ_{-I}) = \int_M \cdots \int_M \tau^k(m_1, \ldots, m_I, \mathbf{m}^\circ_{-I}) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

where $\mathbf{x}^\circ_{-I} = (x^\circ, \ldots, x^\circ)$ and $\mathbf{m}^\circ_{-I} = (m^\circ, \ldots, m^\circ)$.

The buyer's selection strategy $\pi(\gamma, \mu_p)$ induces the probability $z(\pi(\gamma, \mu_p))(x)$ that she either has her type below x as a participant of γ or selects the other sellers whose mechanism is μ_p, similar to (3):

$$z(\pi(\gamma, \mu_p))(x) = 1 - \int_x^{\bar{x}} \pi(\gamma, \mu_p)(s) dF. \tag{21}$$

Let $\mathcal{O}$ be the set of all optimal strategies (c, π) for communicating with the deviating seller and selecting a seller in a continuation equilibrium of the subgame defined by (γ, μ_p) for all $\gamma \in \Gamma$. Following (4) and (5), we can derive the reduced-form direct mechanism, $\{Q^1_{\pi,c,\mu_p}, \ldots, Q^K_{\pi,c,\mu_p}, T_{\pi,c,\mu_p}\}$ from $\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ and $z(\pi(\gamma, \mu_p))$. Then, it is straightforward to show that $\mu_{c,\mu_p}(\gamma)$ is Bayesian incentive compatible (BIC) for any $(c, \pi) \in \mathcal{O}$: For all $x, x' \in X$,

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k_{\pi,c,\mu_p}(x) + T_{\pi,c,\mu_p}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k_{\pi,c,\mu_p}(x') + T_{\pi,c,\mu_p}(x').$$

[12] (γ, μ_p) means that the deviating principal's mechanism is γ, whereas every non-deviating principal's mechanism is μ_p.

Note that the Bayesian incentive compatibility of $\mu_{c,\mu_p}(\gamma)$ depends on the mechanism μ_p chosen by the other sellers because the buyer's communication and selection strategies depend on mechanisms chosen by both sellers.

Following (6) and (8), we can then use the reduced-form direct mechanism to derive (i) the interim expected payoff, denoted by $U_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi, x)$, for the buyer of type x associated with selecting the deviating seller with γ and (ii) the ex-ante expected payoff, denoted by $\Phi_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi)$, for the deviating seller with γ. Our next theorem is the key result in this section.

Theorem 2 *Any equilibrium (DIC) allocation in a competing mechanism game relative to G_D is also supported in an equilibrium of a general competing mechanism game relative to G_Γ.*

We prove Theorem 2 by two steps. We start with some basics. Let Ω be the set of all direct mechanisms. Let $\mathcal{O}_c$ be the projection of $\mathcal{O}$ onto the space of the buyer's communication strategies. Then, let $\mathcal{M}(\mu_p)$ be the set of *all incentive compatible direct mechanisms* that can be induced from all mechanisms in Γ that a deviating seller might choose in a continuation equilibrium given the non-deviating seller's DIC direct mechanism $\mu_p \in \Omega_D$:

$$\mathcal{M}(\mu_p) := \left\{ \mu_{c,\mu_p}(\gamma) \in \Omega : \forall c \in \mathcal{O}_c,\ \forall \gamma \in \Gamma \right\}. \tag{22}$$

Let $\Pi(\mu, \mu_p)$ be the set of all optimal strategies of selecting a seller in a continuation equilibrium where the deviating seller chooses an incentive compatible direct mechanism μ directly from $\mathcal{M}(\mu_p)$. We can then establish the following lemma.

Lemma 2 *Given $\mu_p \in \Omega_D$, the following equality holds:*

$$\sup_{\gamma \in \Gamma} \left(\sup_{(c,\pi) \in \mathcal{O}} \Phi_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi) \right) = \sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu,\mu_p) \in \Pi(\mu,\mu_p)} \Phi_J(\mu, \mu_p, \pi) \right). \tag{23}$$

Proof. Given μ_p that each non-deviating seller chooses, suppose that the deviating seller's mechanism is $\gamma \in \Gamma$. Given (γ, μ_p), let each buyer communicate with the deviating seller upon selecting him and select a seller according to $c(\gamma, \mu_p)$ and $\pi(\gamma, \mu_p)$ for some $(c, \pi) \in \mathcal{O}$. Since $\mathcal{O}$ is the set of all possible communication and selection strategies in a continuation upon a seller's deviation to a mechanism in Γ, $\mu_{c,\mu_p}(\gamma) \in \mathcal{M}(\mu_p)$ and further it is a continuation equilibrium that each buyer uses $\pi(\mu_{c,\mu_p}(\gamma), \mu_p) = \pi(\gamma, \mu_p)$ to select a seller and truthfully report her type to the seller who she selects when the deviating seller directly chooses $\mu_{c,\mu_p}(\gamma)$. This implies (23) because $\mathcal{M}(\mu_p)$ is a subset of Ω, which is embedded into Γ. ■

Lemma 2 implies that an equilibrium DIC allocation (μ^*, π^*) in a general competing mechanism game relative to G_D is also an equilibrium of a general competing mechanism game relative to G_Γ if

$$\Phi_J(\mu^*,\mu^*,\pi^*) \geq \sup_{\mu\in\mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu,\mu_p)\in\Pi(\mu,\mu_p)} \Phi_J(\mu,\mu_p,\pi) \right) \tag{24}$$

We can then use the BIC-DIC equivalence to establish (24).

Corollary 2 *Suppose that a deviating seller offers $\tilde{\mu} \in \mathcal{M}(\mu_p)$, whereas each non-deviating seller uses μ_p to punish the deviator and that it is a continuation equilibrium that buyers chooses the deviator according to $\tilde{\pi}(\tilde{\mu},\mu_p)$ given $(\tilde{\mu},\mu_p)$. Then, there exists a DIC direct mechanism $\mu \in \Omega_D$ such that given (μ,μ_p), it is a continuation equilibrium that buyers choose the deviator according to $\pi(\mu,\mu_p) = \tilde{\pi}(\tilde{\mu},\mu_p)$ and*

$$\begin{aligned} U_J(\mu,\mu_p,\pi,x) &= U_J(\tilde{\mu},\mu_p,\tilde{\pi},x) \ ,\forall x\in X & (25)\\ \Phi_J(\mu,\mu_p,\pi) &= \Phi_J(\tilde{\mu},\mu_p,\tilde{\pi}). & (26)\end{aligned}$$

Proof. The proof is almost the same as the proof of Corollary 1. The only difference is that, with respect to the deviating seller's point of view, the probability that either a buyer's type is less than x or she selects a non-deviator is calculated based on $\pi(\mu,\mu_p)$ and $\tilde{\pi}(\tilde{\mu},\mu_p)$, that is, $z(\pi(\mu,\mu_p))(x)$ and $z(\tilde{\pi}(\tilde{\mu},\mu_p))(x)$. ■

To complete the proof of Theorem 2, first note that (15) and (20) imply that a equilibrium DIC allocation (μ^*,π^*) in a competing mechanism game relative to G_D satisfies

$$\Phi_J(\mu^*,\mu^*,\pi^*) \geq \sup_{\mu\in\Omega_D} \left(\sup_{\pi(\mu,\mu_p)\in\Pi(\mu,\mu_p)} \Phi_J(\mu,\mu_p,\pi) \right)$$

Because $\Omega_D \subset \mathcal{M}(\mu_p)$, it is clear that

$$\sup_{\mu\in\mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu,\mu_p)\in\Pi(\mu,\mu_p)} \Phi_J(\mu,\mu_p,\pi) \right) \geq \sup_{\mu\in\Omega_D} \left(\sup_{\pi(\mu,\mu_p)\in\Pi(\mu,\mu_p)} \Phi_J(\mu,\mu_p,\pi) \right) \tag{27}$$

(26) in Corollary 2 means that (27) holds with equality and hence (24) is satisfied for any equilibrium allocation in a general competing mechanism game relative to G_D. This implies that any equilibrium allocation in a general competing mechanism game relative to G_D is also supported in an equilibrium of a general competing mechanism game relative to G_Γ.

Applying the BIC-DIC equivalence in Corollary 1 to any BIC equilibrium allocation, we can derive the following implications from Theorem 2.

Theorem 3 Φ_J^e *is the complete set of a seller's ex-ante payoffs associated with all equilibrium allocations that are supportable with punishment carried out by a DIC direct mechanism in any general competing mechanism game G_Γ.*

5.1 Characterization without BIC-DIC equivalence

We focus on a robust equilibrium where non-deviators punish a deviator with DIC direct mechanisms. Given a DIC direct mechanism $\mu_p \in \Omega_D$ that non-deviators use, a deviating seller only needs to consider direct mechanisms that are incentive compatible conditional on $\mu_p \in \Omega_D$, that is those in $\mathcal{M}(\mu_p)$. Because of the BIC-DIC equivalence, a deviating seller only needs to consider DIC direct mechanisms, Ω_D. This makes the lower bound of a seller's robust equilibrium ex-ante expected payoff supportable with DIC punishment equal to his minmax value over DIC direct mechanisms.

Even when the BIC-DIC equivalence does not hold, we can consider robust equilibrium BIC allocations supportable with DIC punishment. In that case, the lower bound of a seller's robust equilibrium ex-ante expected payoff supportable with DIC punishment is

$$\inf_{\mu \in \Omega_D} \sup_{\mu' \in \mathcal{M}(\mu)} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right),$$

where $\mathcal{M}(\mu)$ is the set of all BIC direct mechanisms available for a seller conditional on the other sellers' DIC direct mechanism $\mu \in \Omega_D$. The upper bound is a seller's payoff associated with a BIC allocation that is derived by a joint profit maximization.

5.2 Heterogenous buyers and sellers

Given ex-ante homogenous buyers and sellers, we focus on a symmetric equilibrium where sellers use the same pure strategies. This is mainly for notational simplicity. We can relax the homogeneity assumption with heterogenous parameters in the linear payoff structures for buyers and sellers, different numbers of action alternatives for sellers, and different probability distributions for buyers.

For endogenous frictions, we still want to impose symmetry in buyers' selection strategies in that they select sellers with equal probability if their mechanisms are the same. Given this symmetric selection behavior, the main results go through in the sense that the greatest lower bound of a principal's robust equilibrium payoff supportable with DIC punishment is his minmax value over DIC direct mechanisms.[13] The upper bound is not well defined since it is not clear how to set up the joint profit maximization for heterogenous sellers.

6 Applications

Each buyer has a unit demand for a product. If she consumes the product and pays p, her utility is $x - p$, where x is her valuation that follows a probability distribution F over $X = [0, 1]$. There are N buyers who are looking for the product. Each buyer's reservation utility is zero. There are J sellers. Sellers and buyers are all risk neutral.

[13] The results are available upon request. The notation is considerably heavier.

6.1 Competing prices

We consider the case where sellers can produce homogeneous products at a constant marginal cost, normalized to zero, without capacity constraint. Each seller's reservation profit is zero.

Theorem 4 *Suppose that the hazard rate, $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x.*

1. *The set of a seller's ex-ante expected profit that can be supported in a robust equilibrium with DIC punishment of a competing mechanism game relative to G_Γ is $\Phi_J^e = \Phi_J^* = [0, \overline{\phi}_J]$ with*

$$\overline{\phi}_J = \frac{N}{J} x^*(1 - F(x^*)), \tag{28}$$

where x^* is uniquely defined as the value of x satisfying $x - \frac{1-F(x)}{f(x)} = 0$.

2. *Any $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$ can be supported in a robust equilibrium where each seller posts a deviation-reporting contract g^* that offers a single price contingent on buyer's messages $(h_1, \ldots, h_N) \in H^N$ with $H = \{0, 1\}$ such that*

$$g^*(h_1, \ldots, h_N) := \begin{cases} 0 & \text{if } |\{i : h_i = 1\}| > N/2, \\ x(\phi) & \text{otherwise,} \end{cases} \tag{29}$$

where $x(\phi)$ satisfies $\phi = \frac{N}{J} x(1 - F(x))$.

Proof. See Appendix B. ■

In the seller's joint profit maximization, all sellers post an identical direct mechanism $\mu = \{q, p\}$ such that mappings $q : X^N \to [0, 1]$ and $p : X^N \to \mathbb{R}_+$ specify the probability that a buyer receives the product and her payment to the seller respectively (Note that $X = [0, 1]$ so that, if a buyer does not choose, her type message is regarded as zero and $q(0, \mathbf{x}_{-i}) = p(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$). Because every seller's mechanism is identical, a buyer of any type x selects a seller with $\pi(\mu, \mu)(x) = 1/J$.[14] For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ. One can then derive a reduced-form direct mechanism $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ similar to (4) and (5), based on $z(\pi)(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ in (9).

Because a seller can produce each unit at a constant cost and each buyer's valuation is i.i.d., the seller's joint profit maximization problem is to find $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ that maximize

$$N \int_0^1 P(x) dz(\pi)(x) \tag{30}$$

[14] If a buyer does not choose any seller, it is equivalent to choosing a seller with equal probability and sending the zero type message.

subject to the incentive compatibility and individual rationality conditions. We can apply the standard monopolist's analysis (e.g., p. 265 in Fudenberg and Tirole 1991) to find out a solution for joint profit maximization. Given (9), we can show that the virtual valuation of a buyer with valuation x is

$$x - \frac{1 - z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1 - F(x)}{f(x)}.$$

Given the monotone hazard rate on $x \in [0, 1]$, it is then clear that the product must be sold only to a buyer whose virtual valuation is equal to zero or higher with probability one. A buyer's payment is then

$$P(x) = xQ(x) - \int_0^x Q(s)ds = \begin{cases} x^* & \text{if } x \geq x^*, \\ 0 & \text{otherwise.} \end{cases} \tag{31}$$

This induces the seller's ex-ante expected profit in the joint profit maximization as (28). Furthermore, any feasible ex-ante profit for a seller $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$ can be expressed as $\phi = \frac{N}{J} x(\phi)(1 - F(x(\phi)))$ for some $x(\phi) \in [0, 1]$ as shown in Appendix B. Therefore, any feasible ex-ante profit ϕ for a seller can be supported in a robust equilibrium where each seller's deviation-reporting contract assigns either zero (constant marginal cost) or $x(\phi)$, depending on buyers' messages on whether or not a competing seller deviates.

6.1.1 Fixed entry cost

Given the number of buyers N, we have assumed that there is a fixed number of sellers in the market. However, we can endogenize the number of sellers in the market. Suppose that a seller has to incur a fixed cost, $C > 0$, to enter the market.

Note that a seller's *gross* ex-ante expected profit $\phi \in \Phi_J^*$ is an equal split of $Nx(1 - F(x))$ as shown in (58) in Appendix B. Assume that $\frac{N}{2} x^*(1 - F(x^*)) - C \geq 0$.[15] Recall that $\frac{N}{J} x(1 - F(x))$ is maximized at $x = x^*$ by excluding buyers whose valuation is less than x^*. Given the monotone hazard rate on x, a sellers' net ex-ante expected profit $\frac{N}{J} x(1 - F(x)) - C$ is increasing in x before $x = x^*$ and decreasing in x after. Furthermore, it is equal to $-C$ at $x = 0$ or 1. Therefore, for all J such that

$$\frac{N}{J} x^*(1 - F(x^*)) - C \geq 0,$$

we can then derive the threshold of the price $x_J^- \leq x^*$ and $x_J^+ \geq x^*$ that satisfy

$$\frac{N}{J} x_J^-(1 - F(x_J^-)) - C = \frac{N}{J} x_J^+(1 - F(x_J^+)) - C = 0. \tag{32}$$

Proposition 1 *Given the number of buyers N,*

[15] This assumption implies that there will be at least two sellers in the market.

1. *the number of sellers in the market in a robust equilibrium with DIC punishment and a fixed cost C of a competing mechanism game relative to G_Γ is unique and equal to $J_c \in \max\left\{J : x_J^- \leq x^*\right\}$.*

2. $\Phi_{J_c}^* = \left[0, \frac{N}{J_c}\bar{R} - C\right]$ *is the complete set of a seller's ex-ante profits that can be supported in a robust equilibrium, where $\bar{R} = x^*(1 - F(x^*))$.*

3. $[x_{J_c}^-, x_{J_c}^+]$ *is the range of prices that can be supported in a robust equilibrium.*

Proof. Because x_J^- is increasing in J, $\max\left\{J : x_J^- \leq x^*\right\}$ is a singleton so that we have a unique J_c.[16] Suppose that the number of sellers in the market is $J < J_c$. If one additional seller enters the market, then we have $J + 1 \leq J_c$ and a seller's profit net of fixed cost in the market with $J + 1$ sellers is as low as zero and as high as $\frac{N}{J+1}\bar{R} - C > 0$. Therefore, it is *(weakly) dominant* for a potential seller to enter the market. We assume that if a seller is indifferent between staying out of the market and entering it, he enters the market. A seller will enter the market as long as $J < J_c$. If $J = J_c$, then $\frac{N}{J+1}\bar{R} - C < 0$. This implies that a seller's profit net of fixed costs in the market with $J+1$ sellers is negative. No seller will enter the market if the number of sellers in the market is J_c. Therefore, the unique number of firms in the market is J_c.

Let ϕ^C denote a seller's ex-ante expected net profit. If $\phi^C = 0$, $x = x_{J_c}^-$ or x_J^+ according to (32). If $\phi^C = \frac{N}{J_c}\bar{R} - C$, $x = x^*$. Further, $\frac{N}{J_c}x(1 - F(x)) - C$ is continuous and increasing in x before $x = x^*$ and decreasing in x after. Therefore, for any given $\phi^C \in \Phi_{\bar{J}}^*$, there exists a price $x(\phi^C)$ such that

$$\frac{N}{J_c}x(\phi^C)(1 - F(x(\phi^C))) - C = \phi^C.$$

For any $\phi^C \in \Phi_{J_c}^*$, we can then use a deviation-reporting contract defined in (29) to sell the product at price $x(\phi^C)$ on the equilibrium path but at zero price off the path. This also shows that the range of equilibrium prices is $[x_{J_c}^-, x_{J_c}^+]$ ■

The intuition of Proposition 1 is clear. If the number of sellers J is less than J_c, a potential seller finds it always to enter the market because he can ensure that his ex-ante expected net profit will be at least as high as his reservation profit, i.e., $\phi^C \in \Phi_{J_c}^*$. If the number of sellers in the market is $J = J_c$, there are already too many sellers in the sense that if one additional seller enters the market, the maximum ex-ante expected profit associated with monopoly price x^* is negative (i.e., $\frac{N}{J_c+1}\bar{R} - C < 0$). Therefore, the number of sellers in the market is uniquely determined and it is equal to J_c.

While the number of sellers in the market is uniquely determined, their ex-ante expected net profit can be any level in $\Phi_{\bar{J}}^* = \left[0, \frac{N}{J_c}\bar{R} - C\right]$ in the finite market. We

[16] Equivalently, $J_c \in \min\left\{J : x_J^+ \geq x^*\right\}$ is unique because x_J^+ is decreasing in J.

are interested in how the ranges of net profits and prices change as the number of buyers increases.

Proposition 2 *As $N \to \infty$, a seller's equilibrium ex-ante expected profit with a fixed entry cost C is uniquely determined and is equal to zero, and the equilibrium price is also uniquely determined and is equal to the monopoly price, x^*.*

Proof. Let s be the ratio of the number of buyers to the number of sellers, i.e., $s = \frac{N}{J}$. Define the ratio s^* that satisfies

$$s^* \bar{R} - C = 0 \tag{33}$$

Given the number of buyers N, J_c is indeed equal to $\lfloor N/s^* \rfloor$, which denotes the largest integer that does not exceed N/s^*. Therefore the upper bound of a seller's ex-ante expected profit is

$$\frac{N}{\lfloor N/s^* \rfloor} \bar{R} - C \tag{34}$$

Since $\lim_{N\to\infty} \frac{N}{\lfloor N/s^* \rfloor} = s^*$, the upper bound of a seller's expected profit converges to zero as $N \to \infty$. Because $x_{J_c} = x_{\lfloor N/s^* \rfloor}$ is the price that makes a seller's ex-ante expected profit equal to zero as shown in (32), this implies that

$$\lim_{N\to\infty} x_{\lfloor N/s^* \rfloor} = x^*.$$

Because $x_{\lfloor N/s^* \rfloor}$ is the upper bound of the equilibrium price, the monopoly price is the unique equilibrium price as $N \to \infty$. ■

In equilibrium, each seller serves the equal proportion of the market in expectation, which is the ratio of the number of buyers to the number of sellers. There is a unique ratio s^* that makes each seller's expected gross profit equal to his fixed entry cost in joint profit maximization. With a finite number of buyers, the equilibrium ratio of buyers to sellers may not the same as that unique ratio. Proposition 2 shows that as the number of buyer increases, the equilibrium ratio of buyers to sellers converges to it, resulting in the monopoly price but leaving zero net profits with every seller. This large market result holds even when each seller faces a capacity constraint. We see this in the auction environment examined in the next subsection

6.2 Competing auctions

In Section 6.1, each seller has no capacity constraint. What if each seller can produce at most a single unit? The literature has studied competing auctions where sellers are restricted to choose their reserve prices in the (second price) auction (Burguet and Sákovics 1999, Peters 1997, Peters and Severinov 1997, Virag 2010). It shows that there is a pure-strategy equilibrium of the competing auction game in a *large market*

(Peters 1997, Peters and Severinov 1997) where the number of buyers and sellers goes to infinity given a fixed ratio of buyers to sellers and that it is robust in that no seller can gain by deviating to any arbitrary mechanism (Han 2015). In this equilibrium, every seller sets reserve price equal to his cost of producing the product. While the literature has found one robust equilibrium, it is not yet known if there are additional robust equilibria.

Theorem 3 below identifies the upper bound of a seller's ex-ante expected profit, $\overline{\phi}_J$ in a robust equilibrium with DIC punishment of a competing mechanism game relative to G_Γ when sellers can produce at most a single unit at the constant cost of zero.

Proposition 3 *Suppose that the hazard rate $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x. Then $\overline{\phi}_J$ is reached when all sellers offer an auction with reserve price x^* such that*

$$x^* - \frac{1-F(x^*)}{f(x^*)} = 0 \tag{35}$$

Each seller's ex-ante expected profit is

$$\overline{\phi}_J = \frac{N}{J}\int_{x^*}^{1}\left(x - \frac{1-F(x)}{f(x)}\right)\left(1 - \frac{1-F(x)}{J}\right)^{N-1} f(x)dx. \tag{36}$$

Let s be the ratio of the number of buyers to the number of sellers, i.e., $s = \frac{N}{J}$. In a large market, it becomes

$$\overline{\phi}_\infty := \lim_{J\to\infty}\overline{\phi}_J = s\int_{x^*}^{1}\left[\left(x - \frac{1-F(x)}{f(x)}\right)e^{-s(1-F(x))}\right] f(x)dx. \tag{37}$$

Proof. See Appendix C. ■

$\overline{\phi}_J$ is a seller's ex-ante expected profit in the joint profit maximization where every seller uses an identical selling mechanism. Therefore, each buyer with valuation x selects a seller with equal probability $\pi(\mu,\mu))(x) = 1/J$ and hence the probability that a buyer whose valuation is less than x or she selects another seller is given by

$$z(\pi(\mu,\mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x.$$

Then, a buyer's virtual valuation is

$$x - \frac{1-z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1-F(x)}{f(x)}.$$

Given the monotone hazard rate, the virtual valuation is increasing and the optimal mechanism takes the form of an auction with reserve price: The object goes to the highest valuation buyer if it is sold at all. It is sold if and only if the highest valuation among participating buyers is no less than x^*. Appendix C shows that (36) and (37) are a seller's ex-ante expected profits in finite and large markets respectively.

The lower bound of a seller's robust equilibrium ex-ante expected profit $\underline{\phi}_J$ is defined as his minmax value over all possible DIC direct mechanisms according to its definition in (13). However, it is difficult to derive $\underline{\phi}_J$ when there are both limited liability and capacity constraint as in the auction environment. Here, we rather focus on the set of a seller's ex-ante expected profits that can be supported in a robust equilibrium where non-deviating sellers punish the deviator by changing the reserve price in their auctions. The lower bound of a seller's ex-ante expected profits is then expressed in terms of the minmax value with respect to reserve prices in the auctions.

Let $\mu(x)$ denote an auction with reserve price x. Then, $\Phi_J(\mu(x'), \mu(x), \pi)$ denote a seller's expected profit when he posts an auction with reserve price x' given that $J-1$ sellers all post auctions with reserve price x and buyers select a seller according to their selection strategy π.

Peters (1997) shows that in a large market, a deviating seller cannot do better with any arbitrary direct mechanism than she does with an auction, given any distribution of reserve prices of auctions chosen by non-deviating sellers. We can apply the result in Han (2015) to show that in a large market, a deviating seller cannot do better with any arbitrary mechanism than he does with a direct mechanism, given any distribution of reserve prices of auctions chosen by non-deviating sellers. Therefore, we have that, for all $x \in [0,1]$,[17]

$$\lim_{J\to\infty}\left[\sup_{\gamma\in\Gamma}\left(\sup_{(c,\pi)\in\mathcal{O}}\Phi_J(\gamma,\mu(x),c,\pi)\right)\right] = \lim_{J\to\infty}\left[\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x),\pi)\right]. \tag{38}$$

This implies that a deviating seller only needs to consider an auction with reserve price even if he can offer any arbitrary selling mechanism. Importantly, (38) holds independent of the non-decreasing property of the buyer's virtual valuation. This is the key to the robustness in a large market. Let

$$\underline{\phi}_\infty := \lim_{J\to\infty}\left[\min_{x\in[0,1]}\left(\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x),\pi)\right)\right].$$

Proposition 4 $\tilde{\Phi}_\infty = \left[\underline{\phi}_\infty, \overline{\phi}^\infty\right]$ *is the range of a seller's ex-ante expected profits that can be supported in a robust equilibrium of a competing mechanism game relative to* G_Γ*, where non-deviating sellers punish a deviating seller by changing their reserve prices.*

Proof. Let $x^p \in [0,1]$ be a reserve price such that

$$\lim_{J\to\infty}\left[\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x^p),\pi)\right] = \underline{\phi}_\infty.$$

[17]For any given array of auctions offered by sellers, there exists a unique selection strategy π in a continuation equilibrium (Peters and Severinov 1997, Peters 1997, Virag 2010). This is why the maximum over the selection strategies is not taken.

Then, we have

$$\lim_{J\to\infty} \Phi_J(\mu(x^p), \mu(x^p), \pi) \leq \lim_{J\to\infty} \left[\max_{x'\in[0,1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \underline{\phi}_\infty \tag{39}$$

by the definition of the maximum operator. On the other hand, we have that

$$\lim_{J\to\infty} \Phi_J(\mu(x^*), \mu(x^*), \pi) = \overline{\phi}^\infty \geq \lim_{J\to\infty} \left[\max_{x'\in[0,1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \underline{\phi}_\infty \tag{40}$$

Furthermore, when every seller's reserve price is the same, a seller's ex-ante expected profit,

$$\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi) = s \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx',$$

is continuous in x. Given the continuity of $\lim_{J\to\infty} [\Phi_J(\mu(x), \mu(x), \pi)]$, (39) and (40) imply that there exists at least one x such that

$$\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi) = \underline{\phi}_\infty.$$

Furthermore, given the non-decreasing hazard rate and the definition of x^*, (a) $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-increasing in x over $[x^*, 1]$ while it reaches its maximum at $x = x^*$ and its minimum, $\lim_{J\to\infty} \Phi_J(\mu(1), \mu(1), \pi) = 0$ at $x = 0$ in the range of $[x^*, 1]$, and (b) $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-decreasing in x over $[0, x^*]$, while it reaches its maximum at $x = x^*$ and its minimum at $x = 0$ in the range of $[0, x^*]$. Because $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is continuous in x, (a) and (b) imply that there exists at least one $x(\phi)$ such that $\lim_{J\to\infty} \Phi_J(\mu(x(\phi)), \mu(x(\phi)), \pi) = \phi$ for all $\phi \in \tilde{\Phi}_\infty$.

For any given $\phi \in \tilde{\Phi}_\infty$, each seller posts the deviation-reporting contract

$$g^*(h_1, \ldots, h_N) := \begin{cases} \mu(x^p) & \text{if } |\{i : h_i = 1\}| \geq \frac{N}{2} \\ \mu(x(\phi)) & \text{otherwise} \end{cases}. \tag{41}$$

Given this contract, buyers truthfully report whether or not the competing seller deviates to all sellers, and also their true valuation upon selecting a seller. Because $\phi \geq \underline{\phi}_\infty$, (38) implies that a seller cannot gain by deviating to any arbitrary mechanism in a large market. ■

For any ex-ante expected profit level ϕ, we can identify at least one reserve price $x(\phi)$ that induces each seller's ex-ante expected profit equal to ϕ when every seller offers an auction with $x(\phi)$. It is then straightforward to maintain the profit level ϕ through the deviation-reporting contract that makes the reserve price as a function on reports by buyers on whether or not a competing seller deviates, as specified in (41).

6.2.1 Fixed entry cost

Suppose that any seller can enter the market but has to incur a fixed entry cost of C. We can then endogenize the equilibrium ratio of buyers to sellers.

Proposition 5 *Consider a robust equilibrium of a competing mechanism game relative to G_Γ, where non-deviating sellers punish a deviating seller by changing their reserve prices and any seller can freely enter the market at the cost of C. The unique equilibrium selling mechanism is the auction with reserve price x^*. The seller's equilibrium ex-ante expected net profit is zero in that the equilibrium ratio of the number of buyers to the number of sellers, s^*, satisfies*

$$s^* \in \left\{ s : s \int_{x^*}^{1} \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx' - C = 0 \right\}. \tag{42}$$

Proof. If the current ratio of buyers to sellers s' induces a negative ex-ante expected net profit with an auction with reserve price x^*, there is no equilibrium where the seller's ex-ante expected net profit is non-negative. This is because the seller's ex-ante expected net profit is jointly maximized by auctions with reserve price x^*. This implies that some sellers in the market will leave the market, so that it is not an equilibrium.

Suppose that the current ratio of buyers to sellers s' induces a positive ex-ante expected profit net of the fixed cost at reserve price x^*, i.e.,

$$s' \int_{x^*}^{1} \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx' - C > 0. \tag{43}$$

Because the left hand of (43) is continuous in s', there exists $\epsilon > 0$ such that

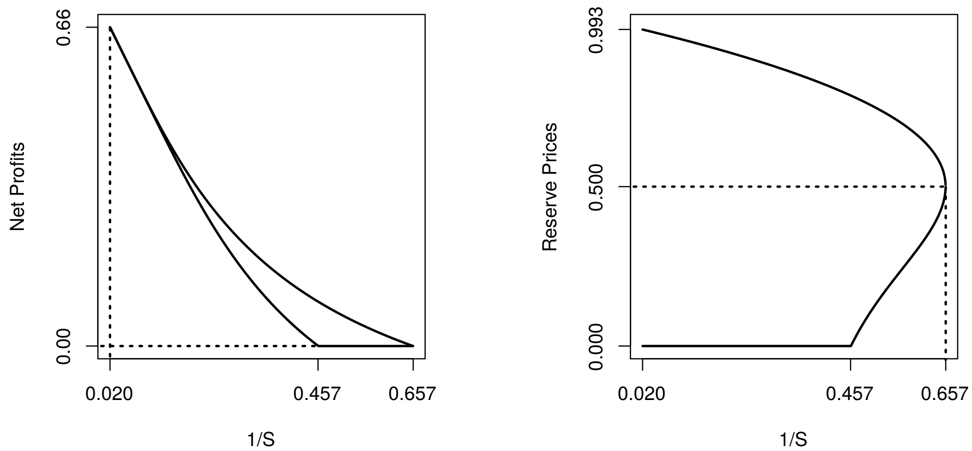
$$(s' - \epsilon) \int_{x^*}^{1} \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-(s-\epsilon)(1-F(x'))} \right] f(x')dx' - C > 0 \tag{44}$$

Let $s = s' - \epsilon$ given arbitrary $\epsilon > 0$ that satisfies the inequality above. Given $s < s'$, we construct all possible equilibria with a deviation-reporting contract ($s < s'$ implies that there are now more sellers given the number of buyers). Note that given the non-decreasing hazard rate, the seller's ex-ante expected profit,

$$s \int_{x}^{1} \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx',$$

is non-decreasing in x over $[0, x^*]$ and non-increasing in x over $[x^*, 1]$. Furthermore, the seller's ex-ante expected profit is zero at $x = 1$. This implies that there exists at least one x in $(x^*, 1)$ such that s

$$s \int_{x}^{1} \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx' - C = 0$$



(a) Net profits

(b) Reserve prices

Figure 1: Ranges of Eq. Profits and Reserve Prices

Let $X(s)$ be the set of reserve prices that make a seller's ex-ante expected profit net of the fixed cost equal to zero:

$$X(s) := \left\{ x : s \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx' - C = 0 \right\} \tag{45}$$

Define $x^+(s) := \max X(s)$ and

$$x^-(s) := \begin{cases} 0 & \text{if } s \int_0^1 \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx' - C \geq 0 \\ \min X(s) & \text{otherwise} \end{cases}$$

Whether $x^-(s) = 0$ or not depends on the seller's net ex-ante expected profit associated with zero reserve price. If it is not negative, $x^-(s) = 0$ because one cannot further lower the reserve price. If it is positive, then $x^-(s)$ will be higher than zero.

The complete set of reserve prices that induce non-negative ex-ante expected profit net of the fixed cost is

$$[x^-(s), x^+(s)].$$

Given the new ratio of buyers to sellers, s, every seller in the market's equilibrium ex-ante expected net profit is as low as zero and as high as (44). Therefore, it is (weakly) dominant for a potential seller to enter the market. This implies that s^* must satisfy (42) and the reserve price must be x^*. Once s^* satisfies (42), no seller has an incentive to leave or enter the market. ■

The example below shows the ranges of a seller's ex-ante expected profit net of the entry cost and reserve price as the ratio of sellers to buyers ($1/s$) changes given parametric specifications on each buyer's valuation and the fixed entry cost.

Example 1 *Each buyer's valuation independently follows the uniform distribution on* $[0, 1]$ *and each seller can produce at most one unit of the good at zero cost: However, a seller has to incur the fixed cost of 0.3 to enter the market.*

We numerically derive the minmax value of a seller's ex-ante expected profit over reserve prices at all possible s and it is reached when the seller sets his reserve price equal to zero and also the other sellers' reserve prices are zero. This is true in both the infinite case and the finite case with a large number of sellers and buyers: For the finite case, we fix the number of buyers to 10,000 and we start the number of sellers at 100 and increase it by 100 to vary s. This minmax value of a seller's ex-ante expected profit is the seller's maximum profit, after he enters the market, that he can get upon his unilateral deviation given others' deviation-reporting contract.

Figure 1 shows the numerical results for the ranges of equilibrium reserve prices and ex-ante profit net of the fixed cost as we increase $1/s$ (i.e., the ratio of the number of sellers to the number of buyers).[18] The minmax value of a seller's ex-ante expected profit net of the fixed cost given $1/s$ is reached when his reserve price is equal to zero given that other sellers' reserve prices are equal to zero.

The left panel shows the range of equilibrium net profits. The upper bound is a seller's jointly maximized profit at the reserve price, 0.5. The lower bound is the maximum between zero and the minmax value of a seller's profit net of the fixed cost. When $1/s$ is very low (i.e., there is a very small number of sellers relative to the number of buyers), the minmax value of a seller's profit net of the fixed cost is very high even though every seller's reserve price is zero. This is because each seller is expected to have a buyer with valuation very close to zero. This is why the lower bound and the upper bound of the equilibrium net profit are virtually identical at $1/s = 0.02$. The net minmax value starts decreasing as $1/s$ increases from $1/s = 0.02$. When $1/s \approx 0.457$, the net minmax value is exactly equal to zero. As $1/s$ further increases, the net minmax value becomes negative so that the lower bound of a seller's equilibrium net profit is supported with only a reserve price higher than zero. The range of equilibrium profits shrinks and eventually collapses to zero at $1/s^* \approx 0.657$ (i.e., $s^* \approx 1.523$),[19] where every seller's reserve price is the monopoly reserve price, 0.5.

The right panel shows the ranges of equilibrium reserve prices. The horizontal dotted line corresponds to the reserve price equal to 0.5, which is the level of reserve

[18] We derive the ranges of equilibrium reserve prices and ex-ante expected net profits in both (a) the finite market with $N = 10000$ and J being increased by 100 from the initial number 100 and (b) its infinite case with the corresponding s. The results are virtually identical.

[19] To be exact, a seller's ex-ante expected profit is 0.00002536 when every seller chooses the monopoly reserve price 0.5 given $N = 10000$ and $J = 6566$. One more seller makes a seller's ex-ante expected profit negative even when every seller chooses the monopoly reserve price 0.5.

price that jointly maximizes each seller's ex-ante expected profit. The curve above the dotted line shows the upper bound of the reserve prices and the one below shows the lower bound of the reserve prices. Any reserve prices between the two are supportable in equilibrium. For the lower bound, note that, after $1/s \approx 0.457$, the net minmax value of a seller's ex-ante expected profit is negative when every seller's reserve price is zero. Therefore, every seller's reserve price must be positive to have zero net profits as the equilibrium net profit. This is why the lower bound is flat before $1/s \approx 0.457$ and keeps increasing after $1/s \approx 0.457$. The range of the equilibrium reserve prices eventually converges to the monopoly reserve price, 0.5, as $1/s$ approaches 0.657.

7 Concluding Remarks

This paper proposes and studies general competing mechanism games of incomplete information, one for markets with frictions and the other for markets without frictions. The approach taken in this paper can be used in various applications for the directed search model such as competing prices, competing auctions and competition in on-line markets. We believe that our approach can also be applied to other problems. For example, Norman (2004) considers public good provision with exclusion for a single mechanism designer. Using the results from this paper we can also consider competing public good provisions with exclusion where buyers eventually select one seller for a public good.

This paper is based on the private value environment in the sense that each buyer's type affects only her payoffs. It is not difficult to imagine an interdependent value environment where a buyer's payoff depends on other buyers' types as well. In this case, one can consider a set of ex-post incentive compatible (EPIC) direct mechanisms (Bergemann and Morris 2005) to punish a deviator. The property of ex-post incentive compatibility also does not depend on the endogenous distribution of the number of participating buyers given that the participating buyers report their true types. Therefore, one can always fix truthful type reporting to non-deviators. Even without the BIC-EPIC equivalence, we can consider robust equilibrium BIC allocations supportable with EPIC punishment: The lower bound of a seller's robust equilibrium ex-ante expected payoff supportable with EPIC punishment is

$$\inf_{\mu\in\Omega_E} \sup_{\mu'\in\mathcal{M}(\mu)} \left(\sup_{\pi(\mu',\mu)\in\Pi(\mu',\mu)} \Phi_J(\mu',\mu,\pi) \right),$$

where Ω_E is the set of all EPIC direct mechanisms and $\mathcal{M}(\mu)$ is the set of all possible BIC direct mechanisms available for a seller conditional on the other sellers' EPIC mechanism $\mu \in \Omega_E$. The upper bound is a seller's payoff associated with a BIC allocation that is derived by a joint profit maximization.

Appendix A. Proof of Corollary 1

Consider any feasible allocation $(\tilde{\mu}, \tilde{\pi}) \in Z$. Therefore, it is a continuation equilibrium that each buyer of type x chooses each seller with equal probability $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) = 1/J$ whenever $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) > 0$ and reports her true type upon selecting a seller. Then, a seller's ex-ante expected payoff and a buyer's interim expected payoff, $\Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi})$ and $U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x)$, are derived with the type distribution $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))(x)$.

Let $\tilde{Q}^k$ be the reduced form of $\tilde{q}^k$ in $\tilde{\mu} = \{\tilde{q}^1, \ldots, \tilde{q}^K, \tilde{t}\}$ based on $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))$. Let Q^k be the reduced form of q^k in $\mu = \{q^1, \ldots, q^K, t\}$ based on $z(\pi(\mu, \mu))$ with $\pi(\mu, \mu) = \tilde{\pi}(\tilde{\mu}, \tilde{\mu})$. Then, we define $\tilde{V}(x) := \sum_{k \in \mathcal{K}} b^k \tilde{Q}^k(x)$ and $V(x) := \sum_{k \in \mathcal{K}} b^k Q^k(x)$. If $\pi(\mu, \mu) = \tilde{\pi}(\tilde{\mu}, \tilde{\mu})$, then

$$z(\pi(\mu, \mu)) = z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu})) \tag{46}$$

Because $z(\pi(\mu, \mu))$ is a probability distribution over $\bar{X}$, and $\bar{X}$ is the message space for a buyer in a direct mechanism, (46) implies that the probability distribution over $\bar{X}$ for a direct mechanism is preserved and that they are independent of each other. Let $\mathbb{E}[\cdot|z(\pi(\mu, \mu))]$ be the expectation operator over a buyer's type x given the probability distribution $z(\pi(\mu, \mu))$. Given the linear payoff structure, we can apply Theorem 1 and Lemma 3 in Gershkov, et al. (2013) for the case of anonymous (and hence non-discriminatory) mechanisms to show the existence of a DIC direct mechanism μ such that

$$V(x) = \tilde{V}(x) \text{ for all } x, \tag{47}$$

$$\mathbb{E}[Q^k(x)|z(\pi(\mu, \mu))] = \mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \text{ for all } k \in \mathcal{K} \tag{48}$$

with the transfers t, which preserves each buyer's interim expected payoff upon selecting the seller. That is,

$$U_J(\mu, \mu, \pi, x) = U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x) \text{ for all } x \tag{49}$$

Note that (48) is the "*ex-ante*" probability that alternative k is chosen. Taking the expected value of each side of (49) over x and applying (47) yields

$$\begin{aligned} &\mathbb{E}[T(x)|z(\pi(\mu, \mu))] \\ &= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \\ &+ g^k \left(\sum_{k \in \mathcal{K}} \mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] - \sum_{k \in \mathcal{K}} \mathbb{E}[Q^k(x)|z(\pi(\mu, \mu))] \right) \\ &= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))] \end{aligned} \tag{50}$$

The first and the second equalities in (50) hold because of (47) and (48) respectively.

On the other hand, the seller's ex-ante expected payoff associated with μ satisfies

$$\begin{aligned}
&\Phi_J(\mu, \mu, \pi) \qquad (51)\\
&= \sum_{k\in\mathcal{K}} (a^k w + y^k)\mathbb{E}[Q^k(x)|z(\pi(\mu,\mu))] - N \times \mathbb{E}[T(x)|z(\pi(\mu,\mu))]\\
&= \sum_{k\in\mathcal{K}} (a^k w + y^k)\mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))] - N \times \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))]\\
&= \Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi})
\end{aligned}$$

where the second equality holds because of (48) and (50). Therefore, the seller's ex-ante expected payoff is preserved. For any ϕ in Φ_J^*, we can find a (BIC) allocation $(\tilde{\mu}, \tilde{\pi})$ that supports it. (51) implies that it can be supported by a DIC allocation, i.e., $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$.

Because monetary transfer to a participating buyer is restricted to be non-positive, we need to show whether a DIC direct mechanism μ also has the property of "non-positive" monetary transfer to a participating buyer given that the original BIC direct mechanism $\tilde{\mu}$ has that property. Note that given $\tilde{\mu} = \{\tilde{q}^1, \ldots, \tilde{q}^K, \tilde{t}\}$, we have

$$\tilde{T}(\underline{x}) = \int_{\underline{x}}^{\bar{x}} \cdots \int_{\underline{x}}^{\bar{x}} \tilde{t}(\underline{x}, s_2, \ldots, s_I) dz(\pi(\tilde{\mu},\tilde{\mu}))(s_2) \ldots dz(\pi(\tilde{\mu},\tilde{\mu}))(s_I) \leq 0,$$

because monetary transfers to a participating buyer are non-positive and that

$$\tilde{V}(\underline{x}) = \sum_{k\in\mathcal{K}} b^k \tilde{Q}^k(\underline{x}) \geq 0$$

by $b^k \geq 0$ and $\tilde{Q}^k(\underline{x}) \geq 0$ for all k. Applying (46) as the underlying probability distribution over the message space in the direct mechanism to Theorem 1 in Gershkov, et al. (2013), monetary transfers in $\mu = \{q^1, \ldots, q^K, t\}$ are determined by

$$t(x, \mathbf{x}_{-1}) = \frac{\rho(\underline{x}, \mathbf{x}_{-1})}{\tilde{V}(\underline{x})}\tilde{T}(\underline{x}) + \rho(\underline{x}, \mathbf{x}_{-1})\underline{x} - \rho(x, \mathbf{x}_{-1})x + \int_{\underline{x}}^{x} \rho(s, \mathbf{x}_{-1}) ds,$$

where ρ is defined as

$$\rho(s, \mathbf{x}_{-1}) := \sum_{k\in\mathcal{K}} b^k q^k(s, \mathbf{x}_{-1}) \geq 0$$

for all $s \in X = [\underline{x}, \bar{x}]$. Because μ is DIC, $\rho(s, \mathbf{x}_{-1})$ is non-decreasing in s.

First of all, we have

$$t(\underline{x}, \mathbf{x}_{-1}) = \frac{\rho(\underline{x}, \mathbf{x}_{-1})}{\tilde{V}(\underline{x})}\tilde{T}(\underline{x}) \leq 0. \qquad (52)$$

because $\tilde{T}(\underline{x}) \leq 0$, $\tilde{V}(\underline{x}) \geq 0$ and $\rho(\underline{x}, \mathbf{x}_{-1}) \geq 0$.[20] Secondly, consider $t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1})$ for any $x, x' \in X$ with $x > x'$:

$$t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1}) = -\rho(x, \mathbf{x}_{-1})x + \rho(x', \mathbf{x}_{-1})x' + \int_{x'}^{x} \rho(s, \mathbf{x}_{-1}) ds \leq 0. \qquad (53)$$

[20] As in Gershkov, et al (2013), 0/0 is interpreted as 1.

The inequality holds because we have

$$\rho(x, \mathbf{x})x - \rho(x', \mathbf{x})x' \geq \int_{x'}^{x} \rho(s, \mathbf{x})ds$$

given that $\rho(s, \mathbf{x}_{-1})$ is non-decreasing in s with $\rho(s, \mathbf{x}_{-1}) \geq 0$ for all $s \in X \subset \mathbb{R}_+$. (52) and (53) imply that $t(x, \mathbf{x}_{-1}) \leq 0$ for all $x \in X$ given any $\mathbf{x}_{-1}$.

Finally we can reach the following conclusion. Because $(\tilde{\mu}, \tilde{\pi})$ is an allocation where it is a continuation equilibrium that buyers report their true type to a seller who they select according to $\tilde{\pi}$, (μ, π) is an allocation such that (i) it is a continuation equilibrium that buyers report their true type to a seller who they select according to π and (ii) each buyer's interim expected payoffs and the ex-ante expected payoff for the seller with the original direct mechanism $\tilde{\mu}$ are preserved.

Appendix B: Proof of Theorem 4

First, we prove that $\overline{\phi}_J$ is that which is specified in (28). $\overline{\phi}_J$ is derived by the seller's joint profit maximization as in (10). Since all sellers post an identical direct mechanism $\mu = \{q, p\}$ in the joint profit maximization as their selling mechanism and that each buyer selects a seller according to $\pi(\mu, \mu)(x) = 1/J$, we can then derive a reduced-form direct mechanism $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ similar to (4) and (5), based on $z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ according to (9). For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ.

Then, the seller's problem is to find $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ that maximize (30) subject to

$$\begin{aligned} \text{(IC) } xQ(x) - P(x) &\geq xQ(x') - P(x') \text{ for all } (x, x') \in [0, 1], \\ \text{(IR) } xQ(x) - P(x) &\geq 0 \text{ for all } x \in [0, 1]. \end{aligned}$$

Let the interim expected utility for a buyer with valuation $x_i \in [0, 1]$ be denoted by $U(x_i) = x_i Q(x_i) - P(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payment:

$$\int_0^1 \cdots \int_0^1 x_i q(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \int_0^1 U(x_i) dz(\pi)(x_i) \tag{54}$$

By using the envelope theorem, we can show that

$$U(x_i) = U(0) + \int_0^{x_i} Q(s)ds. \tag{55}$$

Because the seller does not want to leave any unnecessary rents to buyers, we have $U(0) = 0$ at the optimum. Substituting (55) into (54) and integrating by parts yields

$$\begin{aligned} & N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \quad (56) \\ = \; & N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N), \end{aligned}$$

The last equality holds because

$$\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.$$

Since the incentive compatibility for buyer i is equivalent to (62), and by the monotonicity of $Q(x_i)$, the optimal mechanism maximizes (56) subject to $q(x_i, \mathbf{x}_{-i}) \geq 0$ and $\sum_{i=1}^{N} q(x_i, \mathbf{x}_{-i}) \leq 1$ for all $(x_i, \mathbf{x}_{-i}) \in X^N$, and $Q(\cdot)$ is non-decreasing.

Because there is no capacity limit, (56) is maximized if the seller sells its products to all buyers whose valuation is no less than x^*, that is, for all $\mathbf{x}_{-i}$

$$q(x_i, \mathbf{x}_{-i}) := Q(x_i) = \begin{cases} 1 & \text{if } x_i \geq x^*, \\ 0 & \text{otherwise.} \end{cases} \quad . \tag{57}$$

Therefore, (56) at the solution for the joint profit maximization is

$$\overline{\phi}_J = \frac{N}{J} \int_{x^*}^{1} \left[\left(x - \frac{1 - F(x)}{f(x)} \right) \right] f(x) dx$$

Given (57), a buyer's payment in the optimal selling mechanism becomes (31).

The optimal selling mechanism $\mu = \{Q, P\}$, characterized by (57) and (31), is DIC because the probability and payment only depends on the buyer's own type message but not other buyers'. It is also deceptively simple to be implemented: A seller simply posts a single price x^*. Any buyer who wants the product purchases it at price x^*.

To complete the proof, let us show how to support a seller's ex-ante expected profit $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$ in an equilibrium. When each seller sells his product at price x and buyers choose each seller with equal probability, a seller's ex-ante expected profit is

$$\phi = \frac{N}{J} x(1 - F(x)). \tag{58}$$

If $x = 0$ or 1, then $\phi = 0$. If $x = x^*$, then $\phi = \overline{\phi}_J$. By the continuity of (58), then for any $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$, there exists at least one $x(\phi)$ that satisfies (58)

Fix any $\phi \in \Phi_J^*$ as a seller's ex-ante expected profit. Let every seller post a deviation-reporting contract g^* specified in (29) with $H = \{0, 1\}$. If no seller deviates from g^*, all buyers report $h = 0$ to every seller whose price is $x(\phi)$. Then, buyers with valuation $x(\phi)$ or higher chooses some seller with equal probability and buys the product at price $x(\phi)$. This continuation equilibrium behavior yields the ex-ante expected payoff of ϕ for every seller.

Suppose that a seller deviates from g^*. Then, all buyers report $h = 1$ to every non-deviating seller whose price goes down to zero. A deviating seller must offer an (expected) price that is no higher than zero in order to attract any buyers. This is clearly not profitable because $\phi \geq 0$. Therefore, there is no profitable deviation to any arbitrary mechanism.

Appendix C: Proof of Theorem 3

$\overline{\phi}_J$ is reached when sellers jointly maximize their ex-ante expected profits. Let a direct mechanism μ be characterized by $\{q_i, p_i\}_{i=1}^N$, where $q_i : X^N \to [0,1]$ and $p_i : X^N \to \mathbb{R}_+$ specify the probability of acquiring the object and the payment to the seller, respectively. Specifically, for all $\mathbf{x} = [x_1, \ldots, x_N]$ and all $i = 1, \ldots, N$, $q_i(\mathbf{x})$ and $p_i(\mathbf{x})$ are the probability that buyer i acquires the object and buyer i's payment to the seller with $q_i(0, \mathbf{x}_{-i}) = p_i(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$.

Because every seller offers an identical direct mechanism, each buyer selects each seller with equal probability, $1/J$. Then, the probability that a buyer's valuation is less than x_i or she selects another seller is given by

$$z(\pi(\mu,\mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x. \tag{59}$$

Because each seller offers identical direct mechanisms, we can fix $z(\pi(\mu,\mu))(x_i)$ to (59) in the seller's joint profit maximization problem. For simplicity, we define

$$z(\pi)(x_i) = z(\pi(\mu,\mu))(x_i)$$

for all μ and all x_i. Then, we can derive the reduced-form direct mechanism $\{Q_i, P_i\}_{i=1}^N$ similar to (4) and (5) based on $z(\pi(\mu,\mu))$ specified in (59). It follows that the seller's joint profit maximization problem is to find a direct mechanism that maximizes her ex-ante expected profit:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N p_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \tag{60}$$

subject to

$$\begin{aligned} \text{(IC) } x_i Q_i(x_i) - P_i(x_i) &\geq x_i Q_i(x_i') - P_i(x_i') \text{ for all } (x_i, x_i') \in [0,1], \\ \text{(IR) } x_i Q_i(x_i) - P_i(x_i) &\geq 0 \text{ for all } x_i \in [0,1], \end{aligned}$$

and $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$.

Let buyer i's interim expected utility be denoted by $U_i(x_i) = x_i Q_i(x_i) - P_i(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payments:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N x_i q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \sum_{i=1}^N \int_0^1 U_i(x_i) dz(\pi)(x_i) \tag{61}$$

By using the envelope theorem, we can show that

$$U_i(x_i) = U_i(0) + \int_0^{x_i} Q_i(s) ds. \tag{62}$$

Because the seller does not want to leave any unnecessary rents to buyers, we have $U_i(0) = 0$ at the optimum. Substituting (62) into (61) and integrating by parts yields

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \tag{63}$$

$$= \int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N), \tag{64}$$

The last equality holds because

$$\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.$$

Because the incentive compatibility for buyer i is equivalent to (62), and by the monotonicity of $Q_i(x_i)$, the optimal mechanism maximizes (64) subject to $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$, and $Q_i(\cdot)$ is non-decreasing.

Because a mechanism is anonymous ($Q_i(\cdot) = Q(\cdot)$), the optimal mechanism takes the form of an auction with reserve price: If the virtual valuation, $x_i - \frac{1-F(x_i)}{f(x_i)}$, is non-decreasing, the object goes to the highest valuation buyer if it is sold at all. It is sold if and only if

$$\max_{i \in \{1, \ldots, N\}} x_i \geq x^*, \text{ with } x^* \text{ defined in (35).}$$

Because the auction is anonymous, the seller's maximum ex-ante expected profit is

$$\begin{aligned} \overline{\phi}_J &= N \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) Q(x) z'(\pi)(x) dx \\ &= \frac{N}{J} \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) \left(1 - \frac{1 - F(x)}{J} \right)^{N-1} f(x) dx. \end{aligned}$$

As we take its limit, $\overline{\phi}_\infty = \lim_{J \to \infty} \overline{\phi}_J$ is equal to the expression in (37).

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