COMPETITIVE INVESTMENTS AND MATCHING: HEDONIC PRICING PROBLEMS

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Abstract

The rational expectations equilibrium in this paper endogenizes the worker’s characteristic decision and the firm’s decisions on wage and job amenity in a large matching economy where the worker’s matching benefits may depend in an arbitrary fashion on the worker’s characteristic, wage, and job amenity. This paper provides the sufficient condition for the unique rational expectations equilibrium and shows step by step how to derive job amenities, workers’ characteristics, and wages in equilibrium. The results suggest that the estimates of compensating wage differentials from the hedonic model would be biased even if the worker’s characteristic and job amenity were fully observable.

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1 Introduction

Cole, Mailath, and Postlewaite (2001) and Peters and Siow (2002) showed that because higher attributes induce a higher likelihood of forming a trading
relationship with a trading partner who also has higher attributes, traders on both sides of a two-sided matching market might have incentives to differentiate themselves from others through costly investment in attributes. For example, a married couple may be concerned about joint labor income determined by their human capital. Since a spouse with higher human capital can earn more labor income, everyone in the marriage market prefers a spouse with higher human capital. Therefore, both men and women may become involved in competitive human capital investment prior to marriage so that they can marry a person with higher human capital (Peters and Siow 2002). When monetary match surplus between a firm and a worker shows the complementarity in a firm’s machinery and a worker’s human capital, firms and workers may also engage in competitive investment in their machinery and their human capital. The reason is that better machinery and higher human capital can generate a higher monetary match surplus, which leads to higher monetary rewards to both traders in equilibrium (Cole, Mailath, and Postlewaite 2001).

This paper develops a solution concept for a competitive matching equilibrium in the general environment where the trader’s matching benefits may depend arbitrarily on the trader’s own attribute, the trading partner’s attribute, and money. The generalization of a competitive matching equilibrium in this dimension is important in light of both theory and application. Theoretically, the efficiency result of the Walrasian equilibrium does not depend on how economic agents value various consumption bundles as long as their valuation satisfies the monotonicity assumption. In fact, the efficiency results stem from the fact that market prices are adjusted until any externalities associated with production and consumption decisions are fully internalized into market prices. This is exactly how Peters and Siow (2002) and Cole, Mailath, and Postlewaite (2001) achieved efficient investments in their competitive matching equilibria. So, their efficiency results should hold in a much more general environment where the trader’s matching benefits may depend in an arbitrary fashion on the trader’s own attribute, the trading partner’s attribute, and money.

In applied works the hedonic model (Rosen 1974, 1986) has been extensively used for pricing the heterogeneous goods. In terms of job matching, workers receive benefits from both wage and job amenity when they work for firms.\(^1\) The hedonic model endogenizes job amenity in a perfectly com-

\(^1\)For example, an academic researcher cares about the research environment in which
petitive setting where firms decide their job amenities and workers decide which job to choose, taking wages as given. The equilibrium wage function reflects the worker’s willingness to pay for a better job amenity (compensating wage differentials). While the hedonic model (Rosen 1974, 1986) implicitly assumes that workers’ production-related characteristics are exogenously given, it is not difficult to imagine that workers’ decisions on characteristic acquirement will in general depend on their expectations on the distribution of wages and job amenities in the future job market. In order to analyze the more comprehensive link between compensating wage differentials, matching, and characteristic investments of all workers and firms, we need an environment where matching benefits to a worker generate a trade off between wage and job amenity: Such a trade off is absent in Peters and Siow (2002) and Cole, Mailath, and Postlewaite (2001) because matching benefits are one-dimensional: either spouse’s human capital or monetary reward, but not both.

When workers invest in their characteristics \((x)\), they should consider the impact of the distribution of workers’ characteristics on firms’ decisions. When they make decisions on wages and job amenities, firms should also consider the impact of the distribution of wages \((w)\) and job amenities \((y)\) on workers’ characteristic decisions. The potential difficulty in firms’ decisions is that when workers have different characteristics, they may have different marginal rates of substitution of job amenity for wage, so the willingness to pay for a better job amenity may differ across workers. This implies that workers with different characteristics may rank jobs differently.\(^2\) The key point of the rational expectations equilibrium proposed in this paper is that what matters for a worker are matching benefits as a whole that the worker receives from both wage and job amenity. In the rational expectations equilibrium everyone has the belief \((v(x))\) about the worker’s matching benefits as a function of the worker’s characteristic. Given this belief, firms choose

an academic institute has invested over time. Lawyers, medical doctors, and consultants are concerned about how their expertise can fit into what law firms, hospitals, consulting companies have specialized in and how much they can increase their expertise by working for a potential employer. Hawng, Mortensen, and Reed (1998) and Lang and Majundar (2004) introduce search and matching frictions in the hedonic model to show wage dispersions.

\(^2\)This difficulty does not arise in Cole, Mailath, and Postlewaite (2001) and Peters and Siow (2002). The reason is that in their models people care about either the monetary transfer from the potential partner or the potential partner’s one-dimensional non-monetary characteristic, but not both.
wages and job amenities and workers choose their characteristics. The belief is adjusted until it is exactly realized in equilibrium.

A worker might want to reduce his characteristic investment just a little bit for any given wage and job amenity offered to him since investments are costly. However, when there are many competing workers there are competitors whose investment levels are arbitrarily close to his. So, any discrete reduction in investments will make him unable to match with the potential firm who is supposed to match with him in equilibrium. This is also true for a firm when she decides wage and job amenity. Essentially the equilibrium belief on a worker’s matching benefits internalizes any externalities associated with characteristic decisions, so it enables workers (firms) to capture the full benefits of their investments. Therefore, the rational expectations outcome is efficient.

While the efficiency result of a competitive matching equilibrium seems intuitive, characterizing a competitive matching equilibrium is not an easy task. In equilibrium, the measure of workers with characteristic \( x \) must be the same as the measure of firms who want characteristic \( x \) at each \( x \). Therefore, we must identify the measures of workers and firms at each \( x \) and check whether the market clearing condition holds. I show that a worker with lower investment costs invests more in his characteristic, expecting a higher characteristic would lead to higher matching benefits. The firm chooses \((y, w)\) that satisfies \( v(x) = u(x, y, w) \) when she likes to match with a worker with characteristic \( x \). I show that if the belief \( v(x) = u(x, y, w) \) makes wages weakly submodular in the worker’s characteristic and job amenity, then the firm’s profits associated with matching with a worker become supermodular in the worker’s characteristic and job amenity. This is because the firm’s profits are decreasing in wage and the worker’s output is assumed to be supermodular in the worker’s characteristic and job amenity. This paper finds the conditions that ensure wages satisfying the belief \( v(x) = u(x, y, w) \) are weakly supermodular. Given conditions identified in this paper, a firm with lower costs of providing job amenity provides a better job amenity and a corresponding wage, expecting her job amenity and wage offer make her match with a worker with a better characteristic. What firms and workers believe is fully realized by the assortative matching in \((x, y)\) in equilibrium. When one side of the market, for example the worker side, is longer than the other, the worker who matches with a firm with the worst job amenity is just indifferent between matching with the firm and staying out of the market. This is the low-end match condition in equilibrium. The efficiency property
in every equilibrium match, the assortative matching, and the low-end match condition pin down the unique rational expectation equilibrium.

The rational expectations equilibrium fully endogenizes workers’ characteristic decisions and firms’ job characteristic (wage and job amenity) decisions in an environment where a worker’s matching benefits may depend in an arbitrary fashion on the worker’s characteristic, job amenity, and wage. This paper shows step by step how to derive job amenities, workers’ characteristics, and wages in equilibrium. The difficulty in estimating the effect of equalizing differences is well known because part of a worker’s characteristic and job amenity is not observable (Brown 1982, Duncan and Holmlund 1983, and Hawng, Reed and Hubbard 1992). In conclusion, this paper shows that the worker’s equilibrium characteristic and equilibrium job amenity are perfectly correlated with each other. Therefore, the estimates of compensating wage differentials from the hedonic model would be biased even if the worker’s characteristic and job amenity were fully observable.

2 Preliminaries

I will construct a two-sided matching model and present it in terms of hedonic pricing problems in labor relationships. However, the model is general enough to be applied to other two-sided matching problems such as buyer-seller relationships or merge/acquisition relationships.

A firm opens a job that needs one worker. A worker can take only one job. A worker’s (production-related) characteristic is denoted by \( x \geq 0 \). A wage \( w \in \mathbb{R} \) is a monetary transfer from a firm to a worker. \( y \geq 0 \) is a job amenity that a firm chooses. We call \((y, w)\) a job characteristic. The firm’s payoff function is \( \Pi(x, y, w, \theta) = f(x, y) - w - c(y, \theta) \) when the firm matches with a worker with characteristic \( x \) by offering job characteristic \((y, w)\). \( f(x, y) \) can be thought of as the output produced by a worker with characteristic \( x \) when job amenity is \( y \). If job amenity does not affect the output produced by a worker, then one can use a simpler functional form \( f(x) \).

3 If we interpret \( y \) to be a firm’s physical capital like Cole, Mailath, and Postlewaite (2001), then \( f(x, y) \) is a natural form of a production function.

The worker’s payoff function is \( U(x, y, w, \delta) = u(x, y, w) - e(x, \delta) \) when the worker invests in \( x \) and works for a firm with job characteristic \((y, w)\). \( u(x, y, w) \) are the matching benefits for a worker with characteristic \( x \) who

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\[^{3}\text{If we interpret } y \text{ to be a firm’s physical capital like Cole, Mailath, and Postlewaite (2001), then } f(x, y) \text{ is a natural form of a production function.} \]
works for a firm with job characteristic \((y, w)\): the worker cares about both wage \(w\) and job amenity \(y\). This creates a trade-off between wage and job amenity when a worker chooses a job. The marginal rate of substitution of job amenity for wage represents the trade-off. This is the essence of Rosen’s model (1974, 1986), which generates compensating wage differentials. Note that to make problems more interesting we allow for the possibility that the marginal rate of substitution may depend on the worker’s characteristic. If the worker’s characteristic does not affect the worker’s matching benefits, then we can simply change the matching benefit function to \(r(y, w)\). In this case the marginal rate of substitution is independent of a worker’s characteristic. Alternatively, one can make the worker’s matching benefit function separable with respect to money (that is, \(u(x, y, w) = a(x, y) + q(w)\)) if we want to impose separability between wage and non-monetary characteristics. None of the results of this paper depend on the potential dependence of the marginal rate of substitution on the worker’s characteristic or separability. I assume monotonicity in the payoff functions: \(f_x > 0, f_y > 0, c_y > 0, c_\theta < 0, u_x > 0, u_y > 0, u_w > 0, e_x > 0, \) and \(e_\delta < 0\).

The worker’s characteristic (the firm’s job characteristic) affects not only his (her) own payoffs but also the potential partner’s payoffs. Therefore, there are externalities associated with characteristic decisions. When we consider characteristic decisions in an isolated single match, a worker and a firm may not have the right incentives for efficient characteristic decisions because for any given wage a worker (a firm) will make a characteristic (job amenity) decision that maximizes only his (her) payoffs. For example, a worker’s optimal characteristic is derived by solving \(\max_x [u(x, y, w) - e(x, \delta)]\) for any given \((y, w)\). This is obviously not an efficient characteristic decision because a worker does not consider the benefits that a firm will get from his characteristic decision.

One of my interests is to see if traders have incentives for differentiating themselves in their characteristics through costly investments and how these incentives induce competitive investment prior to matching. So, I allow heterogeneity on the worker side in terms of the costs of characteristic acquirement and heterogeneity on the firm side in terms of the costs of job amenity acquirement. Let \(\Theta = [\underline{\theta}, \overline{\theta}]\) be the set of firms’ types. \(F(A)\) denotes the measure of firms whose types are in the subset \(A \subseteq \Theta\). Let \(\Delta = [\underline{\delta}, \overline{\delta}]\) be the set of workers’ types. \(G(B)\) denotes the measure of workers whose types are in the subset \(B \subseteq \Delta\). I also allow for the possibility that one side may
be longer than the other. I choose the worker side as the possible long side: $G(\Delta) \geq F(\Theta)$.

3 Rational Expectations Equilibrium

The worker’s matching benefits depend on both wage and job amenity. Therefore, there is a trade-off between them. However, it is important to note that what matters for a worker is matching benefits as a whole. When workers choose their characteristics and firms decide their job characteristics, they have a belief about the worker’s matching benefits. Their belief is that $v(x)$ are the matching benefits to a worker with characteristic $x$. Given the belief function $v(\cdot)$, the worker’s problem is to choose $x$ that maximizes $v(x) - e(x, \delta)$. Let $x_s(\delta)$ be a solution to this problem for all $\delta$. A firm believes that if she wants to match with a worker with $x$, she should have job characteristic $(y, w)$ that creates matching benefits $u(x, y, w) = v(x)$. If she wants to match with a worker with characteristic $x$, a firm chooses a job amenity that maximizes $f(x, y) - g(x, y) - c(y, \theta)$, where $g(x, y)$ satisfies $u(x, y, g(x, y)) = v(x)$ for all $(x, y)$. Let $b(x)$ be a solution to this problem. Since the constraint $u(x, y, w) = v(x)$ is incorporated in $g(x, y)$, the firm’s optimal job characteristic is simply $(b(x), g(x, b(x)))$ when she wants to match with a worker with characteristic $x$. Finally, a firm must figure out the worker’s characteristic that she wants the most. Let $S(x, \theta) = f(x, b(x)) - g(x, b(x)) - c(b(x), \theta)$ be the maximum payoffs when a firm hires a worker with characteristic $x$. The firm’s decision is to choose the $x$ that maximizes $S(x, \theta)$. Let $x_d(\theta)$ be a solution to the problem of maximizing $S(x, \theta)$ for all $\theta$. A firm then chooses the optimal job characteristic $(b(x_d(\theta)), g(x_d(\theta), b(x_d(\theta))))$ expecting she could match with a worker with characteristic $x$. The type-$\theta$ firm matches with the type-$\delta$ worker if and only if $x_d(\theta) = x_s(\delta)$.

**Definition 1** \( \{ x_d(\cdot), x_s(\cdot), v(\cdot) \} \) is a rational expectations equilibrium if

1. market clearing condition: \( F[\theta : x_d(\theta) \geq x] = G[\delta : x_s(\delta) \geq x] \) at each $x$.

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4 Alternatively, one can also assume that it is the firm side that is long. The efficiency of competitive matching equilibrium does not depend on which side is long. However, the trader who is indifferent between staying unmatched and matching with the equilibrium partner is on the long side. Section 4 explains how this helps us determine the equilibrium outcome.
2. consistency: \( v(x) = u(x, b(x), g(x, b(x))) \) at each \( x \)

Part 1 of Definition 1 states the market clearing condition. For each characteristic \( x \), we have an implicit market where workers who supply characteristic \( x \) and firms who demand characteristic \( x \). In equilibrium, the measure of workers who supply \( x \) must be equal to the measure of firms who demand characteristic \( x \) at each characteristic \( x \). Given \( v(\cdot) \), each worker simply chooses the characteristic that maximizes his payoffs. Given \( v(\cdot) \), each firm derives an optimal job characteristic at each characteristic \( x \) and then chooses an optimal characteristic \( x \) that maximizes her payoffs. Then, each firm chooses the optimal job characteristic corresponding to the optimal characteristic \( x \). \( v(\cdot) \) is adjusted until it is exactly realized in equilibrium. This is what part 2 of Definition 1 states.

Pareto optimality of the rational expectations equilibrium outcome comes from its definition. Define an implicit function \( q(x, y, w) = v(x) - u(x, y, w) = 0 \) for all \( (x, y, w) \). \( q \) divides the space of \( (x, y, w) \) into two disjoint subspaces. Define \( D = \{ (x, y, w) : q(x, y, w) = 0 \} \), \( D_- = \{ (x, y, w) : q(x, y, w) < 0 \} \), and \( D_+ = \{ (x, y, w) : q(x, y, w) > 0 \} \). When a firm with job characteristic \( (y, w) \) matches with a worker with characteristic \( x \), \( (x, y, w) \) must be in \( D \) because the firm chooses \( (y, w) \) and the worker chooses \( x \) subject to \( q(\cdot, \cdot, \cdot) = 0 \). In equilibrium, any \( (x', y', w') \) that makes a worker strictly better off lies on \( D_- \) and any \( (x', y', w') \) that makes a firm strictly better off lies on \( D_+ \). Since \( D \), \( D_- \) and \( D_+ \) are disjoint, there does not exist an alternative allocation that makes someone strictly better off without making anyone worse off. This argument holds for every match in equilibrium. Therefore, Pareto optimality of the rational expectations equilibrium comes directly from its definition.

Intuitively, a worker may want to reduce his characteristic investment just a little bit for any given wage and job amenity offered to him since investments are costly. However, when there are many competing workers there are competitors whose investment levels are arbitrarily close to his. So, any discrete reduction in his investment will cause him to match with the potential firm who would match with him. This is also true for a firm when she decides wage and job amenity. The belief function \( v(\cdot) \) essentially internalizes any externalities associated with characteristic decisions, so it enables workers (firms) to capture the full benefits of their investments. Therefore, the rational expectations outcome is Pareto optimal.
Equilibrium Properties

Without any restrictions on payoff functions, characterizing a rational expectations equilibrium is not an easy task. In equilibrium, the measure of workers with characteristic $x$ must be the same as the measure of firms who want characteristic $x$ at each $x$. Therefore, we must identify the measures of workers and firms at each $x$ and check whether the market clearing condition is satisfied. I introduce the supermodular (and submodular) assumption in order to sort out equilibrium matching. The supermodular assumptions on payoff functions are imposed as follows: $f_{xy} > 0$, $u_{xy} \geq 0$, and $u_{yw} \geq 0$. The submodular assumptions are imposed as follows: $c_{y\theta} < 0$, $u_{xw} \leq 0$, and $e_{x\delta} < 0$. Note that the supermodular (submodular) assumptions on $u(\cdot, \cdot, \cdot)$ admit weak inequality, so $u(\cdot, \cdot, \cdot)$ admits separability with respect to its arguments. When the supermodular (submodular) assumptions hold with weak inequality, they are called weakly supermodular (submodular).

I also assume that $F$ and $G$ are strictly increasing. This assumption ensures that there is no bunching in equilibrium. In other words, the measures of sellers and buyers who supply and demand $x$ are exactly zero at each $x$. I will focus on a class of well-behaved equilibria in which $v(\cdot)$ is continuous and differentiable. As it will be clear later, the supermodular and submodular assumptions are sufficient for the unique rational expectations equilibrium in a class of well-behaved equilibria. Lemma 2 shows the necessary condition for the equilibrium set of characteristics.

Lemma 2 Any rational expectations equilibrium $\{x_d(\cdot), x_s(\cdot), v(\cdot)\}$ satisfies

1. if $\delta > \delta'$, then $x_s(\delta) > x_s(\delta')$,
2. $x_d(\cdot)$ is increasing in a firm’s type,
3. and if $\theta > \theta'$, then $b(x_d(\theta)) > b(x_d(\theta'))$

Proof. See Appendix 8.1.

Lemma 2 shows that equilibrium matching is assortative in types or alternatively in $(x, y)$. The supermodular and submodular assumptions are sufficient conditions for assortative matching. In general, the assortative matching in characteristics of traders has been playing the central role of analyzing matching behavior. Departing from hedonic pricing problems for
the moment, let $x$ denote a seller’s characteristic and $y$ denote a buyer’s characteristic. Suppose that the monetary matching surplus is $s(x, y)$ when a buyer with characteristic $x$ and a seller with characteristic $y$ match with each other. Becker’s famous result (1973) shows that supermodularity of $s(\cdot, \cdot)$ is the sufficient condition for assortative matching given characteristics on both sides.

Suppose that the cost of providing job amenity and the cost of investment in the worker’s characteristic are incurred prior to matching. When workers and firms go on the market the increases in their payoffs due to matching with someone are represented only by $f(x, y) - w$ for a firm and $u(x, y, w)$ for a worker. We have three arguments that affect the matching benefits. While higher $x$ and higher $y$ are better for both a firm and a worker, the firm’s ranking on wages is exactly opposite to the worker’s ranking. Nonetheless, one might be tempted, for example, to assume the (weak) supermodularity of $u(\cdot, \cdot, \cdot)$ with respect to $(x, y, w)$ (that is, $u_{xy} \geq 0$, $u_{yw} \geq 0$, and $u_{xw} \geq 0$) and the supermodularity of $f(\cdot, \cdot)$, in light of Becker’s sufficient condition for assortative matching, in order to pin down the equilibrium matching.

In equilibrium, firms and workers believe the existence of matching benefit returns on the worker’s investment. By the submodularity of $e(\cdot, \cdot)$, a higher type worker invests more in equilibrium. For any given $(x, y)$, a firm will offer a wage that satisfies $v(x) = u(x, y, w)$. Therefore, the wage depends on $(x, y)$. Recall that $g(x, y)$ denotes the wage satisfying $u(x, y, g(x, y)) = v(x)$ for all $(x, y)$. Note that the firm’s matching benefits are monotonically decreasing in $w$ while the worker’s matching benefits are monotonically increasing. Therefore, the firm’s matching benefit function becomes $f(\cdot, \cdot) - g(\cdot, \cdot)$. Given $f_{xy} > 0$, the firm’s matching benefit function becomes supermodular if $g(\cdot, \cdot)$ is weakly submodular ($g_{xy} \leq 0$). The weak submodularity of $g(\cdot, \cdot)$ is ensured if $u_{xy} \geq 0$, $u_{yw} \geq 0$, and $u_{xw} \leq 0$ (See Appendix). Since the conditions on cross partial derivatives of $u(\cdot, \cdot, \cdot)$ are imposed with weak inequality, those conditions are easily satisfied if, for example, (i) the worker’s matching benefit function is separable with respect to money (that is, $u(x, y, w) = a(x, y) + q(w)$) or (ii) the worker’s matching benefits are independent of his characteristic $x$ (that is, $u(x, y, w) = r(y, w)$). In fact the direction of $g_{xy}$ is indeterminate if we assume the supermodularity of $u(\cdot, \cdot, \cdot)$ with respect to $(x, y, w)$.

The supermodularity of $f(\cdot, \cdot) - g(\cdot, \cdot)$ and the submodularity of $c(\cdot, \cdot)$ make a higher type firm invest more in $y$ and want higher $x$ in equilibrium. Therefore, we have assortative matching in types or alternatively in $x$ and $y$. 


Proposition 3 \( \{x_s(\cdot), x_d(\cdot), v(\cdot)\} \) is a rational expectations equilibrium if and only if it satisfies

1. at each \((\theta, \delta)\) such that \(F([\theta, \theta]) = G([\delta, \delta])\), \(x_s(\delta) = x_d(\theta)\),
2. at \((x, y, w) = (x, b(x), g(x, b(x)))\) for any \(x = x_d(\theta) = x_s(\delta)\)
   \[v(x) = u(x, y, w)\]
3. at \(\delta^*\) such that \(F([\theta, \theta]) = G([\delta^*, \delta])\),
   \[v(x_s(\delta^*)) - e(x_s(\delta^*), \delta^*) = \max_x [u(x, 0, 0) - e(x, \delta^*)]\] (1)
4. for all \(\delta\)
   \[v'(x_s(\delta)) - e_x(x_s(\delta), \delta) = 0\]
5. at \((x, y, w) = (x, b(x), g(x, b(x)))\) for any \(x = x_d(\theta) = x_s(\delta)\),
   \[f_x(x, y) + \frac{u_x(x, y, w) - e_x(x, \delta)}{u_w(x, y, w)} = 0\]
   \[f_y(x, y) - c_y(y, \theta) + \frac{u_y(x, y, w)}{u_w(x, y, w)} = 0\]

Proof. Necessity. Lemma 2 implies that a higher-type firm wants to match with a worker with a better characteristic in equilibrium. Since a higher type firm matches with a higher type worker in equilibrium, the workers who stay out of the market have the types below the types of workers who match with a firm in equilibrium. Therefore, condition 1 must be satisfied in equilibrium. Condition 2 simply restates part 2 of Definition 1. Consider condition 3. Look at the low-end match between the type-\(\theta\) firm and the type-\(\delta^*\) worker, where \(\delta^*\) satisfies \(F([\theta, \theta]) = G([\delta^*, \delta])\). Suppose that the type-\(\delta^*\) worker strictly prefers entering the market with characteristic \(x_s(\delta^*)\) to staying out of the market. Then, workers whose types are arbitrarily close to \(\delta^*\) believe that they would be better off by entering the market with characteristic \(x_s(\delta^*)\), so they will also enter the market with characteristic \(x_s(\delta^*)\). This implies that the total measure of workers in the market is larger than the total measure of firms: the market clearing condition is not satisfied. It also implies that we must have condition 3 in equilibrium. Condition 4
is the first-order condition for the worker’s problem. If this is not satisfied for some type-δ worker, \( x_s(\delta) \) does not maximize the type-δ worker’s payoffs. Therefore, it must be satisfied in equilibrium. Suppose that condition 5 is not satisfied for some match in equilibrium. Then, given the constraint \( q(x) = v(x) - u(x, y, w) \), either a firm or a worker in the match does not maximize her or his payoffs. So, condition 5 must be satisfied in equilibrium.

**Sufficiency.** Condition 4 states the first-order condition for the worker’s problem. Let \( \eta(x) \) be the type of a worker who wants to supply characteristic \( x \). The first-order condition for the worker who wants to supply attribute \( x \) is
\[
\frac{v'(x)}{v(x)} - e(x, \eta(x)) = 0.
\]
The submodularity of \( e(\cdot, \cdot) \) implies that \( v'(x') - e(x', \eta(x)) > 0 \) for all \( x' < x \) and \( v'(x') - e(x', \eta(x)) < 0 \) for all \( x' > x \). It shows that the worker’s payoffs are increasing at characteristics lower than his equilibrium characteristic and decreasing at characteristics higher than his equilibrium characteristic. I will show that Definition 1 is satisfied using conditions 1, 2, and 4 and then show that condition 5 is sufficient for an optimal solution to a firm’s problem. For any \( x \in [x_s(\delta^\ast), x_s(\delta)] \), let \( \varphi(x) \) be the type of a firm who wants characteristic \( x \). This function \( \varphi(\cdot) \) satisfies \( F(\varphi(x), \bar{\theta}) = G(\eta(x), \bar{\delta}) \) by condition 1 for all \( x \in [x_s(\delta^\ast), x_s(\delta)] \). Since different types of firms and workers demand and supply different \( x \) in equilibrium, the measure of firms who demand characteristic \( x \) and the measure of workers who supply characteristic \( x \) are zero respectively at each \( x \). Therefore, \( F(\varphi(x), \bar{\theta}) = G(\eta(x), \bar{\delta}) \) for all \( x \in [x_s(\delta^\ast), x_s(\delta)] \) becomes the market-clearing condition. Using conditions 3 and 4 we can see that \( g_x(x, b(x)) \) is the negative marginal rate of substitution of a worker’s characteristic for wage and \( g_y(x, b(x)) \) is the negative marginal rate of substitution of job amenity for wage. Now consider the firm’s maximization problem. Given the properties of \( g_x(x, b(x)) \) and \( g_y(x, b(x)) \), the first-order condition for a firm who demands characteristic \( x \) becomes
\[
S_x(x, \varphi(x)) = \left[ f_x(x, y) + \frac{u_x(x, y, w) - e_x(x, \eta(x))}{u_w(x, y, w)} \right] + \left[ f_y(x, y) - c_y(y, \varphi(x)) + \frac{u_y(x, y, w)}{u_w(x, y, w)} \right] b'(x).
\]
at \( (x, y, w) = (x, b(x), g(x, b(x))) \). When condition 5 holds, the first-order condition for a firm who demands characteristic \( x \) is zero when we evaluate it at \( x \): \( S_x(x, \varphi(x)) = 0 \). Since \( S(\cdot, \cdot) \) is supermodular (see Appendix), we
have \( S_x(x', \varphi(x)) > 0 \) for all \( x' < x \) and \( S_x(x', \varphi(x)) < 0 \) for all \( x' > x \). It shows that the firm’s payoffs are increasing at the worker’s characteristics lower than the worker’s equilibrium characteristic that the firm wants and decreasing at the worker’s characteristics higher than the worker’s equilibrium characteristic. Note that the firm’s first-order condition incorporates the market-clearing condition and the consistency condition of Definition 1. Finally, condition 3 ensures that no workers whose types are below \( \delta^* \) want to enter the market because of the submodularity of \( e(\cdot, \cdot) \).

5 Existence

The existence of the rational expectations equilibrium characterized in Proposition 3 is equivalent to the existence of the matching benefit function \( v(\cdot) \) such that \( v(x) = u(x, b(x), g(x, b(x))) \). Note that the equilibrium wage \( g(x, b(x)) \) depends on both the worker’s characteristic and job amenity. For notational simplicity let \( m(x) = g(x, b(x)) \), then the existence of \( v(\cdot) \) can be established by showing the existence of \( b(\cdot) \) and \( m(\cdot) \).

Let \( \alpha(\theta) \) be the type of a worker with whom the type-\( \theta \) firm matches in equilibrium. Since a higher type firm matches with a higher type worker in equilibrium, \( \alpha(\cdot) \) satisfies \( F([\theta, \overline{\theta}]) = G([\alpha(\theta), \overline{\theta}]) \) for all \( \theta \). Given \( \alpha(\cdot) \), Proposition 3.4 becomes

\[
\begin{align*}
  f_x(x, b(x)) + \frac{u_x(x, b(x), m(x)) - e_x(x, \alpha(\theta))}{u_w(x, b(x), m(x))} &= 0 \quad (2) \\
  f_y(x, b(x)) - c_y(b(x), \theta) + \frac{u_y(x, b(x), m(x))}{u_w(x, b(x), m(x))} &= 0 \quad (3)
\end{align*}
\]

at each \( \theta \). Given \( \{b(\cdot), m(\cdot)\} \), equations (2) and (3) make us identify the type of a firm who matches with a worker with characteristic \( x \). So, equation (2) provides us with the function \( \phi \) such that \( \theta = \phi(x, b(x), m(x)) \) denotes the type of a firm who matches with a worker with characteristic \( x \) at each \( x \). Similarly, let \( \theta = \tau(x, b(x), m(x)) \) denote the type of a firm identified from equation (3) at each \( x \). In equilibrium, we must have

\[
\phi(x, b(x), m(x)) = \tau(x, b(x), m(x)) \quad (4)
\]

at each \( x \). Equation (4) shows that \( b(\cdot) \) and \( m(\cdot) \) are interdependent in equilibrium. From equation (4) one can derive the function \( b(x) = \gamma(m(x), x) \)
such that \( \phi(x, \gamma(m(x), x), m(x)) = \tau(x, \gamma(m(x), x), m(x)) \) at each \( x \). If we take the derivative of \( b(x) = \gamma(m(x), x) \) with respect to \( x \) we have \( b'(x) = \gamma_w(m(x), x)m'(x) + \gamma_x(m(x), x) \).

The first-order condition for a worker’s problem is \( v'(x) - e(x, \delta) = 0 \). Since we have \( v(x) = u(x, b(x), m(x)) \) in equilibrium, the first-order condition for the worker becomes

\[
\begin{align*}
&u_x(x, b(x), m(x)) + u_y(x, b(x), m(x))b'(x) + \\
&u_w(x, b(x), m(x))m'(x) - e(x, \alpha(\theta)) = 0
\end{align*}
\]

One can finally derive the ordinary differential equation \( m'(x) = h(m(x), x) \) by plugging (i) \( \theta = \tau(x, b(x), m(x)) \), (ii) \( b(x) = \gamma(m(x), x) \), and (iii) \( b'(x) = \gamma_w(m(x), x)m'(x) + \gamma_x(m(x), x) \) into equation (5). The boundary condition is derived by applying Proposition 3.3 and 3.4 to the low-end match \( (\theta, \delta^*) \). Once we derive \( m(\cdot) \) from \( m'(x) = h(m(x), x) \) we can derive \( b(\cdot) \) from \( \hat{b}(x) = \gamma(m(x), x) \).

A solution to \( m'(x) = h(m(x), x) \) exists and it is unique when \( h(\cdot, \cdot) \) is Lipschitzian (Waitman 1986). This condition is ensured when \( h(\cdot, \cdot) \) is bounded. To ensure that \( h(\cdot, \cdot) \) is bounded we need technical restrictions on \( F, G, V, \) and \( U \) since \( h(\cdot, \cdot) \) depends on them. First, we need \( F \) and \( G \) to be continuous and their first-order derivatives to be bounded. Second, \( V \) and \( U \) should be continuous and twice differentiable. Note that \( h(\cdot, \cdot) \) depends not only on the first-order derivatives of \( V \) and \( U \) but also on the second-order (cross) partial derivatives of \( V \) and \( U \) because \( \gamma_w(m(x), x) \) and \( \gamma_x(m(x), x) \) depend on the second-order (cross) partial derivatives of \( V \) and \( U \). Therefore, the first-order and second-order derivatives of \( V \) and \( U \) should be bounded. With the concavity assumption on \( V \) and \( U \), these technical restrictions on \( F, G, V, \) and \( U \) ensure that \( h(\cdot, \cdot) \) is Lipschitzian. Once we derive \( \{b(\cdot), m(\cdot)\} \) the equilibrium matching benefit function for the worker becomes \( v(\cdot) = u(\cdot, b(\cdot), m(\cdot)) \). Although all these technical restrictions are satisfied, one might not get a closed form of the solution when \( F, G, V, \) and \( U \) have complicated forms. In this case we need to numerically solve the ordinary differential equation.

6 Conclusion

We can also think of an alternative solution concept. For example, the notion of stable matching can be applied to derive competitive solutions (Cole,
Mailath, and Postlewaite 2001). That is, we can set up a wage as a function $(w(x, y))$ of job amenity and the worker’s characteristic. For given $w(\cdot, \cdot)$ a worker chooses $(x, y)$ that maximizes $u(x, y, w(x, y)) - e(x, \delta)$ and a firm chooses $(x, y)$ that maximizes $f(x, y) - w(x, y) - c(y, \theta)$. The matching outcome induced by a function $w(\cdot, \cdot)$ is stable if no firm and worker pair can make themselves strictly better off by matching and making a monetary transfer (wage). This approach will generate efficient equilibrium outcomes because $w(\cdot, \cdot)$ internalizes any externalities associated with firms’ job amenity decisions and workers’ investment decisions. However, the notion of stable matching alone would not pin down $(x, y, w(x, y))$ at the low-end match, so there will be a continuum of stable matching equilibria. In this sense one can interpret my solution concept as a selection criterion for stable matching equilibria.

While this paper assumes that job amenity and the worker’s characteristic are one-dimensional respectively, the efficiency result of the rational expectations equilibrium does not depend on this restriction. The efficiency result can be extended into an environment where job amenity and the worker’s characteristic are multidimensional as long as matching benefits satisfy the monotonicity assumption. I made job amenity and the worker’s characteristic one-dimensional purely for the sake of notational simplicity. Multidimensionality of job amenity, the worker’s characteristic, and the type will, however, matter when we try to characterize the rational expectations equilibrium and to demonstrate its existence. The worker’s equilibrium matching benefit function is derived by the market-clearing condition. Since the measure of workers with characteristic $x$ must be equal to the measure of firms who want characteristic $x$, we need to identify these two measures at each $x$. It is not obvious how to do this when job amenity, the worker’s characteristic, and the type are multidimensional.

The only approach available for the multidimensional cases is the one proposed by Levin (1999) in the context of mass transportation problems. Suppose that one needs to transport a continuum of sand from one side to the other side. The objective is to minimize the total transportation cost. Suppose that each load of sand has multidimensional attribute $x$ and each location on the other side has multidimensional attribute $y$. When $x$ is transported to $y$ the cost is $c(x, y)$. Levin shows that when $c$ satisfies a generalized Spence-Mirrlees condition there exists an optimal measure preserving mapping $p : x \mapsto y$ that minimizes the total transportation cost. While it may provide a practical tool for some economic applications, it cannot be ap-
plied in an obvious way to my environment. First, the measures of $x$ and $y$ in mass transportation problems are exogenously given rather than endogenously determined. Second, mass transportation problems do not allow the possibility of leaving some $x$ in the original location. In other words, it excludes the possibility that a worker might not want to match with any firm in equilibrium.

## 7 Appendix

We start with some basics. If a firm wants $x$, the firm’s optimal job characteristic $(y, g(x, y))$ satisfies $u(x, y, g(x, y)) = v(x)$. Take the derivative of this equation with respect to $y$, then

$$u_y(x, y, g(x, y)) + u_w(x, y, g(x, y))g_y(x, y) = 0. \quad (6)$$

Take the derivative of (6) with respect to $x$:

$$u_{yx}(x, y, g(x, y)) + u_{yw}(x, y, g(x, y))g_x(x, y) + u_{wx}(x, y, g(x, y))g_y(x, y) + u_w(x, y, g(x, y))g_{yx}(x, y) = 0.$$ 

Therefore,

$$g_{yx}(x, y) = -\frac{u_{yz}(x, y, g(x, y))}{u_w(x, y, g(x, y))} - \frac{u_{yw}(x, y, g(x, y))g_x(x, y)}{u_w(x, y, g(x, y))} - \frac{u_{wx}(x, y, g(x, y))g_y(x, y)}{u_w(x, y, g(x, y))}.$$ \quad (7)

From the constraint $u(x, y, g(x, y)) = v(x)$ we can derive $g_x(x, y) > 0$ and $g_y(x, y) < 0$. Since $u(\cdot, \cdot, \cdot)$ satisfies the supermodular and submodular assumptions imposed on $u(\cdot, \cdot, \cdot)$, we can conclude $g_{xy} \leq 0$ from equation (7).

Now we look at a property of the firm’s maximum payoff function $S(\cdot, \cdot)$. Suppose that the type-$\theta$ firm matches with a worker with characteristic $x'$ and the type-$\theta'$ firm matches with a worker with characteristic $x$, where $\theta > \theta'$ and $x > x'$. $z(x', \theta)$ and $z(x, \theta')$ are optimal job amenities that the firms offer. Let $\bar{y} \equiv \max[z(x', \theta), z(x, \theta')]$ and $\underline{y} \equiv \min[z(x', \theta), z(x, \theta')]$. Suppose

---

5Carlier and Ekeland (2003) applied the technique for mass transportation problems to optimal city structure problems. Carlier (2003) also applied the technique to a principal-agent problem.
that the type-θ firm chooses job characteristic \((\overline{y}, g(x, \overline{y}))\) in order to match with a worker with characteristic \(x\) and that the type-\(\theta'\) firm chooses a job characteristic \((y, g(x', y))\) in order to match with a worker with characteristic \(x'\). In this case the sum of the two payoffs is \(f(x, \overline{y}) - g(x, \overline{y}) - c(\overline{y}, \theta) + f(x', y) - g(x', y) - c(y, \theta')\). This is strictly greater than \(S(x', \theta) + S(x, \theta')\) because of the supermodularity of \(f(\cdot, \cdot)\) and the submodularity of \(g(\cdot, \cdot)\) and \(c(\cdot, \cdot)\):

\[
S(x', \theta) + S(x, \theta') < f(x, \overline{y}) - g(x, \overline{y}) - c(\overline{y}, \theta) + f(x', y) - g(x', y) - c(y, \theta') \quad \text{(8)}
\]

Since \((\overline{y}, g(x, \overline{y}))\) is only one among many job characteristics that generate the matching utility \(v(x)\), \(S(x, \theta)\) must be weakly greater than \(f(x, \overline{y}) - g(x, \overline{y}) - c(\overline{y}, \theta)\). For the same reason \(S(x', \theta')\) must be weakly greater than \(f(x, \overline{y}) - g(x', y) - c(y, \theta')\). Therefore, we have

\[
f(x, \overline{y}) - g(x, \overline{y}) - c(\overline{y}, \theta) + f(x', y) - g(x', y) - c(y, \theta') \leq S(x, \theta) + S(x', \theta') \quad \text{(9)}
\]

Equations (8) and (9) imply:

\[
S(x', \theta) + S(x, \theta') < S(x, \theta) + S(x', \theta') \quad \text{(10)}
\]

for all \(x > x'\) and all \(\theta > \theta'\). Therefore, \(S(\cdot, \cdot)\) is supermodular.

### 7.1 Proof of Lemma 2

In equilibrium, we have

\[
\begin{align*}
    v(x_s(\delta)) - e(x_s(\delta), \delta) & \geq v(x_s(\delta')) - e(x_s(\delta'), \delta), \\
    v(x_s(\delta')) - e(x_s(\delta'), \delta') & \geq v(x_s(\delta)) - e(x_s(\delta), \delta').
\end{align*}
\]

The sum of two inequalities yields

\[
e(x_s(\delta), \delta) + e(x_s(\delta'), \delta') \leq e(x_s(\delta'), \delta) + e(x_s(\delta), \delta').
\]

This inequality and the submodular assumption on \(e\) imply that if \(\delta > \delta'\), then \(x_s(\delta) \geq x_s(\delta')\). Since \(x_s(\delta')\) is an optimal characteristic for the type-\(\delta'\) worker, it satisfies the first-order condition \(v'(x_s(\delta')) - e_x(x_s(\delta'), \delta') = 0\). This implies that \(v'(x_s(\delta')) - e_x(x_s(\delta'), \delta) > 0\) because of the submodular assumption on \(e\). This shows that \(x_s(\delta) \neq x_s(\delta')\). Therefore, we have that if \(\delta > \delta'\), then \(x_s(\delta) > x_s(\delta')\).
Now I prove the second property. Suppose not, then the type-θ firm matches with a worker with characteristic \( x \) and the type-θ′ firm matches with a worker with characteristic \( x' \), where \( \theta > \theta' \) and \( x < x' \). Let \((y, g(x, y))\) be the type-θ firm’s job characteristic and \((y', g(x, y'))\) the type-θ′ firm’s job characteristic. In equilibrium, the type-θ firm (weakly) prefers job characteristic \((y, g(x, y))\) to job characteristic \((y', g(x', y'))\) and the type-θ′ firm (weakly) prefers \((y', g(x', y'))\) to job characteristic \((y, g(x, y))\):

\[
\begin{align*}
\text{for } & (y, g(x, y)) \\
& f(x, y) - g(x, y) - c(y, \theta) \geq f(x', y') - g(x', y') - c(y', \theta), \\
& f(x', y') - g(x', y') - c(y', \theta') \geq f(x, y) - g(x, y) - c(y, \theta').
\end{align*}
\]

where at least one holds with strict inequality. The sum of the two inequalities is

\[
c(y, \theta) + c(y', \theta') < c(y', \theta) + c(y, \theta'). \tag{11}
\]

We know that \( x' > x \) and \( \theta > \theta' \). If \( y \leq y' \), inequality (11) contradicts the submodular assumption of \( c(\cdot, \cdot) \), so it must be the case that \( y > y' \).

Suppose that the type-θ firm changes her job characteristic to \((y, g(x', y))\) to match with a worker with characteristic \( x' \). In equilibrium this change is not profitable:

\[
\begin{align*}
\text{for } & (y, g(x', y)) \\
& f(x, y) - g(x, y) - c(y, \theta) \geq f(x', y') - g(x', y') - c(y', \theta).
\end{align*}
\]

Suppose that the type-θ′ firm changes her job characteristic to \((y', g(x, y'))\) to match with a worker with characteristic \( x \). In equilibrium this change is not profitable:

\[
\begin{align*}
\text{for } & (y', g(x, y')) \\
& f(x', y') - g(x', y') - c(y', \theta') \geq f(x, y') - g(x, y') - c(y', \theta'). \tag{13}
\end{align*}
\]

The sum of inequalities (12) and (13) is

\[
f(x, y) - g(x, y) + f(x', y') - g(x', y') \geq f(x', y') - g(x', y') + f(x, y') - g(x, y'). \tag{14}
\]

Since \( y > y' \) and \( x' > x \), inequality (14) contradicts the supermodular assumption on \( f(\cdot, \cdot) - g(\cdot, \cdot) \); it implies that \( x_d(\cdot) \) is increasing in the firm’s type.

Finally, I prove the third property. Because of the submodular assumption of \( c(\cdot, \cdot) \), the optimal job amenity is not decreasing in the firm’s type. Let \((x, y)\) be the type-θ firm’s optimal choice of the worker’s characteristic and job amenity. Then the first-order condition for an optimal job amenity
for the type-θ firm satisfies \( f_y(x, y) - g_y(x, y) - c_y(y, \theta) = 0 \). Suppose that \( x' \) is the worker’s characteristic that the type-θ' firm wants, where \( \theta' > \theta \). Since the worker’s characteristic that a firm want is increasing in the firm’s type, we have \( x' > x \). Since \( f(\cdot, \cdot) - g(\cdot, \cdot) \) is supermodular in \((x, y)\) and \( c(\cdot, \cdot) \) is submodular in \((y, \theta)\), we have \( f_y(x', y) - g_y(x', y) - c_y(y, \theta') > 0 \). This implies that the optimal job amenity that the type-θ' firm chooses is strictly higher than \( y \). Therefore, the optimal job amenity offered by a firm is increasing in the firm’s type.

References


