DEPARTMENT OF ECONOMICS

WORKING PAPER SERIES

2014-04 (rev. 10/14)

McMASTER UNIVERSITY

Department of Economics
Kenneth Taylor Hall 426
1280 Main Street West
Hamilton, Ontario, Canada
L8S 4M4

http://www.mcmaster.ca/economics/

Robust Competitive Auctions

Seungjin Han*

October 14, 2014

Abstract

This paper shows that a competitive distribution of auctions (Peters, 1997) is robust to the possibility of a seller's deviation not only to a direct mechanism, but rather to any arbitrary mechanism. It characterizes equilibrium allocations that are not only robust but also independent of market information transmission from buyers to sellers. For this type of equilibrium allocation, one only needs to design a market with a subset of direct mechanisms. In fact, a (constrained) ex-post efficient allocation is implemented by a market information-free robust equilibrium in a market with the set of second price auctions with reserve prices.

JEL classification codes: C71; D82
Keywords: competitive auctions, market information-free property, robust equilibrium, competing mechanism design

1 Introduction

Most optimal auction theory is developed for the monopolistic environment where a buyer's outside options are exogenously given (Myerson, 1981). However, when many sellers auction off very similar products or standardized items, a buyer's outside options are endogenous because she can acquire a similar product or item by entering other auctions. Therefore, we need a different analytical approach for competing auction problems where many sellers compete to sell their products.

*Acknowledgement to be added. Address: Department of Economics, McMaster University, 1280 Main Street West Hamilton, ON, CANADA. Email: hansj@mcmaster.ca

Peters (1997) studies a decentralized auction market with many sellers and buyers, all of whom differ in terms of their valuation of the item being traded. In a decentralized economy, it is natural to expect frictions among buyers; for example, buyers visit sellers with the same probability when their selling mechanisms are identical, and this selection behavior, which arises from an incentive consistent selection strategy, causes frictions that lead to a non-degenerate probability distribution over the number of participating buyers across sellers. Peters' result showed that even with buyers' frictional selection behavior, as the number of sellers increases to infinity, a seller cannot increase his profits by offering any alternative direct mechanism instead of a second-price auction with reserve price equal to his cost. However, it is not yet known whether the competitive distribution of second-price auctions is robust to the possibility that sellers may deviate to any arbitrary mechanism, not just direct mechanisms.[1]

Since McAfee's (1993) paper on competing auctions, it has been noticed that the message space associated with a direct mechanism is not large enough for a buyer to report her private information in a market with many sellers. This is because a buyer is informed not only about her private valuation of the object but also about the mechanisms offered by the other sellers in the market. Therefore, the standard Revelation Principle defined over the buyer's valuation typically fails in the decentralized economy with multiple sellers, even though it forms the basis of the optimal auction theory for a monopolist.[2] Nonetheless, McAfee (1993) argues that a pure-strategy equilibrium in direct mechanisms continues to be an equilibrium in the game with larger strategy spaces for sellers. His logic is that if every seller employs a pure strategy for his mechanism offer, a deviating seller has nothing to learn from asking buyers about other sellers' mechanisms because he believes that other sellers will not deviate.

The logic underlying McAfee's argument, however, is incomplete in two

[1] That is, robust to the possibility that a seller can gain by deviating to an arbitrary mechanism other than the direct mechanism.

[2] Peck (1997) presents the example with multiple sellers where buyers send false type reports to a seller who offers a direct mechanism when others randomize their mechanism offers, and shows that there is no equivalent incentive-compatible direct mechanism for that seller. Epstein and Peters (1999) develop the universal language that can be used as the message space for buyers to report not only their valuation but also mechanisms offered by competing sellers. The complexity of this language reflects how elaborate a mechanism can be because the terms of trade in a seller's mechanism may depend on what agents report on the other sellers' mechanisms, and the terms of trade in the other sellers' mechanisms may depend on others' mechanisms, and so on. This is called "infinite regress" in competing mechanisms.

respects. Of course, buyers' communication with a deviating seller, together with the deviating seller's arbitrary mechanism, induce a direct mechanism. However, buyers' communication with a deviating seller depends on their communication with non-deviating sellers because the latter affects their probabilities of selecting the deviating seller, which affect their payoffs upon selecting the deviating seller. Buyers' communication with non-deviating sellers may in turn depend on what type of a mechanism the deviating seller offers. Therefore, it is not easy to associate any arbitrary mechanism on the part of the deviating seller with a direct mechanism unless we can clearly assign buyers' communication with non-deviating sellers in continuation equilibrium following a seller's deviation to any possible type of mechanism (direct or not).

Even if this is possible, it need not be the case that a seller has no incentive to deviate to any other mechanism than the direct mechanism. The reason is as follows. Even if we focus on incentive consistent continuation equilibrium to highlight market frictions, there may be multiple incentive consistent continuation equilibria at a profile of direct mechanisms (μ_j, μ_{-j}) offered by sellers, but only one of them is realized at (μ_j, μ_{-j}). If seller j is allowed to offer mechanisms other than direct ones, then he may want to deviate to an arbitrary indirect mechanism, given the other sellers' direct mechanisms μ_{-j}, in order to induce the same μ_j in an incentive consistent continuation equilibrium that is not expected to be played when he offers μ_j. In this continuation equilibrium, a buyer's selection strategy would be different and hence, the deviator may gain. Therefore, a seller may well have incentives to deviate to an arbitrary indirect mechanism even in pure-strategy equilibrium.

Let us explain why the competitive distribution of second-price auctions (Peters, 1997) is robust. First of all, if non-deviating sellers offer second-price auctions, we can always fix buyers' truthful valuation reporting as their communication with non-deviating sellers, regardless of a deviating seller's mechanism (direct or not). Therefore, we can ensure that, given buyers' truthful valuation reporting to non-deviating principals, the set of all possible incentive compatible direct mechanisms includes any incentive compatible direct mechanism that can be induced in any continuation equilibrium upon the deviating seller's arbitrary mechanism. Then, one only needs to tackle the multiplicity of continuation equilibria upon a seller's deviation to a direct mechanism in the set of all incentive compatible direct mechanisms that are available for a deviating seller, given the buyer's truthful valuation reporting to non-deviating sellers. In order to do that, it is important to note that any incentive consistent continuation equilibrium upon seller j's deviation to an arbitrary mechanism induces a direct mechanism and the buyer's selection strategy that is associated with it. Therefore, seller j cannot gain by deviating to any arbitrary mechanism if he cannot gain in *every* incentive consistent continuation equilibrium upon offering every possible *incentive compatible direct mechanism*, given the other sellers' mechanisms. This condition is indeed embedded in the notion of a competitive distribution of auctions in Peters (1997). Therefore, a competitive distribution of second-price auctions is robust to the possibility that sellers may deviate to any arbitrary mechanism, not just direct mechanisms.

Of course, a competitive distribution of auctions may not be the only equilibrium mechanisms that are robust to a seller's deviation to any arbitrary mechanism. For example, Yamashita (2010) and Peters and Troncoso-Valverde (2013) consider a non-frictional economy where agents (buyers) can communicate with all principals (sellers). The set of robust equilibrium allocations is in general quite large. This is possible because a seller's equilibrium mechanism changes his terms of trade when the majority of agents send to him the true report on the deviating seller's mechanism. For example, sellers may sustain the monopolistic price by lowering their price down to zero if the majority of buyers send to the non-deviating sellers the true report on a competing seller's price reduction. This will prevent a deviating seller from attracting buyers away from non-deviating sellers in the first place.[3] However, this type of equilibrium allocation that depends on market information transmission from buyers is vulnerable to collusion among buyers. Even when no sellers deviate, buyers can all falsely report to every seller that a competing seller has deviated in order to get a lower price. In this case, the monopolistic price cannot be supported.

Because of potential collusion among buyers when they report market information, this paper proposes the notion of market information-free robust equilibrium that is not only robust but also independent of market information transmission from buyers to sellers in a market with a set of arbitrary mechanisms. The "market information-free" property means that a buyer's communication behavior with a non-deviator off the path upon a seller's deviation to any mechanism is the same as her communication behavior on the path with no deviation by any seller. Therefore, upon some seller's deviation, a buyer sends to a non-deviator a message drawn from the same probability distribution that she uses when no seller deviates. This makes a buyer's

[3]In fact, they used the recommendation mechanism as equilibrium mechanisms where agents actually recommend the direct mechanism that the seller should implement, along with their type report. This has the same effect as agents reporting the deviator's mechanism, along with their type reports because the report on the deviator's mechanism will determine the seller's direct mechanism. See subsection 4.1 for more details.

communication behavior with a non-deviator depend only on her valuation but not on changes in the market. Therefore, regardless of whether there is a deviating seller, a non-deviator's direct mechanism that is induced by buyers' communication given his equilibrium mechanism is always fixed. This implies that a non-deviating seller's terms of trade do not change even when there is a deviating seller.

We characterize market information-free robust equilibrium allocation. Consider an allocation that is implemented by a "market information-free robust equilibrium" in a market with a set of mechanisms. We show that it is also implemented by a market information-free robust equilibrium only with a subset of direct mechanisms in which (i) each seller's equilibrium direct mechanism is incentive compatible not only on the path but also off the path following a seller's deviation to any arbitrary mechanism (market information-free property) and (ii) no seller can gain in any incentive consistent continuation equilibrium upon offering any possible incentive compatible direct mechanism (robustness). Therefore, for a market information-free robust equilibrium allocation, we can focus on direct mechanisms in terms of mechanisms that a market should allow.

The allocation in the competitive distribution of second-price auctions is indeed a market information-free robust equilibrium allocation. Condition (ii) is embedded in the notion itself and condition (i) is, as explained earlier, satisfied because second-price auctions are *dominant strategy incentive compatible*. That is, it is a dominant strategy for a buyer to report her true valuation upon participating in a non-deviating seller's second price auction, regardless of the other buyers' selection probabilities. This makes it possible to fix the distribution of non-deviating sellers' direct mechanisms (i.e., second price auctions), regardless of the deviating seller's mechanism.

An important aspect of competing mechanisms is whether we can design a market that can implement an efficient allocation through a market information-free robust equilibrium. In terms of efficiency, the constrained ex-post efficiency is proper because of lack of coordination. The constrained efficiency means that a seller's object is allocated to the participant with the highest valuation of all the participants who select the seller, including the seller himself. Therefore, one way to achieve this constrained efficiency is for each seller to offer the second price auction with reserve price equal to the seller's cost. The robustness of a competitive distribution of auctions implies that the constrained efficient allocation can be implemented by a market information-free robust equilibrium in a market with a set of mechanisms as small as the set of second price auctions with reserve prices.

2 A Competitive Distribution of Auctions

This section presents the environment for competing auctions in Peters (1997) and his notion of a competitive distribution of auctions. J sellers face κJ buyers: $\mathcal{J} = \{1, \ldots, J\}$ and $\mathcal{I} = \{J+1, \ldots, (\kappa+1)J\}$. Each seller has one unit of an indivisible good to sell. Each buyer needs one unit of the good and the distribution of the buyer's valuation is denoted by F with support $[0, 1]$. Each seller faces a cost of selling his good. The distribution of costs in the population of sellers is denoted by G with support that is contained in $[0, 1]$. In seller j's direct (anonymous) mechanism, the message space for each buyer is $\bar{X} = [0, 1] \cup \{x^\circ\}$, where x° denotes the case where the buyer does not participate in the mechanism. Then seller j's direct mechanism is denoted by $\mu_j = \{p_j, q_j\}$ with the message space $\bar{X}$ for each buyer. Suppose that x is buyer i's message and that $\mathbf{x}$ is the profile of the other buyers' messages. Then $p_j(x, \mathbf{x})$ is the price that buyer i pays to seller j and $q_j(x, \mathbf{x})$ is the probability with which buyer i acquires the object.

Suppose that $\mu = \{\mu_1, \ldots\}$ is the profile of direct mechanisms employed by sellers. We let $\pi(x, \mu_j, \mu_{-j}) \in [0, 1]$ denote the probability with which a buyer with valuation x selects seller j who offers μ_j given the (distribution of) direct mechanisms μ_{-j} offered by other sellers. As explained in subsection 2.1, this formation of the buyer's selection strategy π implies symmetry in continuation equilibrium in the sense that the probability with which a buyer selects a seller depends only on the seller's mechanism but not on the buyer or seller's identity.

Define $z_j(\mu_j, \pi)$ as follows: For all $x \in [0, 1]$,

$$z_j(\mu_j, \pi)(x) = 1 - \int_x^1 \pi(s, \mu_j, \mu_{-j}) f(s) ds, \tag{1}$$

$z_j(\mu_j, \pi)(x)$ is the probability that a buyer's valuation is less than x or that she selects a seller other than j given her selection strategy π.[4] $z_j(\mu_j, \pi)$ then determines $Q_j(x, \mu, \pi)$ and $P_j(x, \mu, \pi)$, where $\mu = (\mu_j, \mu_{-j})$. That is, it determines (i) the reduced-form probability with which a buyer expects to acquire the object from seller j and (ii) the reduced-form price that she expects to pay if she selects seller j's mechanism. Because the mechanism is anonymous and hence, all buyers with the same valuation are treated the same way, we can focus on buyer number $J + 1$ for those reduced forms and

[4] Some of the notation is slightly modified from that which appears in Peters (1997).

hence $Q_j(x,\mu,\pi)$ is given by

$$Q_j(x,\mu,\pi) = \int_0^1 \cdots \int_0^1 q_j(x, s_{J+2}, \ldots s_{(\kappa+1)J}) dz_j(\mu_j,\pi)(s_{J+2}) \ldots dz_j(\mu_j,\pi)(s_{(\kappa+1)J}). \quad (2)$$

We can similarly derive $P_j(x,\mu,\pi)$.

The standard result pertaining to the reduced form mechanism shows that if the seller's mechanism is (Bayesian) incentive compatible given π, then $Q_j(x,\mu,\pi)$ is non-decreasing in x.[5] Given μ, the payoff for a buyer with valuation x who reports her true valuation to seller j is

$$\hat{v}(x,\mu_j,\mu_{-j},\pi) = Q_j(y_j,\mu,\pi)y_j - P_j(y_j,\mu,\pi) + \int_{y_j}^{x} Q_j(s,\mu,\pi)ds, \quad (3)$$

where $y_j = \inf\{x : \pi(x,\mu_j,\mu_{-j}) > 0\}$.

2.1 Incentive Consistent Continuation Equilibrium

Competition among sellers is modelled through a two-stage game. First, sellers offer incentive compatible direct mechanisms. Given an array of incentive compatible direct mechanisms, each buyer selects a seller and reports her valuation to the seller she selects. Frictions in the market are captured in incentive consistent continuation equilibrium.

Definition 1 *The buyer's selection strategy is said to be incentive consistent if it characterizes an incentive consistent continuation equilibrium, that is, for every* $\mu = \{\mu_1, \ldots\}$ *and* x, $\sum_{j=1}^{J} \pi(x,\mu_j,\mu_{-j}) = 1$ *and*

$$\begin{aligned} \pi(x,\mu_j,\mu_{-j}) &= 0 \Longrightarrow \exists \ell \neq j : \hat{v}(x,\mu_\ell,\mu_{-\ell},\pi) \geq \hat{v}(x,\mu_j,\mu_{-j},\pi), \\ \pi(x,\mu_j,\mu_{-j}) &> 0 \Longrightarrow \hat{v}(x,\mu_j,\mu_{-j},\pi) \geq \hat{v}(x,\mu_\ell,\mu_{-\ell},\pi) \ \forall \ell \end{aligned}$$

Because incentive consistency is imposed in the buyer's continuation equilibrium strategy, it implies that no buyer has an incentive to deviate from the selection strategy π when all the other buyers select sellers according to the

[5] The Bayesian incentive compatibility of μ_j depends on each buyer's *endogenous* selection probability $\pi(x,\mu_j,\mu_{-j})$ because it determines the probability distribution $z_j(\mu_j,\pi)$ which, in turn, determines each buyer's (interim) payoff. Therefore, in general one cannot separate the Bayesian incentive compatibility from the buyer's endogenous selection decision.

selection strategy π. Importantly, the incentive consistency imposes symmetry on a buyer's selection strategy in the sense that the probability with which a buyer selects a seller depends only on the seller's mechanism but not on the buyer or seller's identity. In other words, two sellers must be selected with the same probability if those two offer the same mechanism. This captures frictions in the decentralized market due to lack of coordination among buyers.

2.2 Deviation to Direct Mechanisms

As Peters and Severinov (1997) suggest, competing auction games with a finite number of sellers often do not admit a pure-strategy equilibrium.[6] Therefore, Peters (1997) considers a finite approximation of the limit game with an infinite number of sellers and buyers given the fixed ratio κ of buyers to sellers. A cutoff valuation for seller j is the infimum of the set of valuations for which buyers choose seller j with positive probability. Let H denote a distribution of cutoff valuations for the limit game. When H is continuous, $\bar{y}_j$ is the cutoff valuation with the property that the proportion $\frac{j-1}{J}$ of all sellers hold auctions that generate cutoff valuations no higher than $\bar{y}_j$, that is, $\bar{y}_j = \sup\left\{x : H(x) \leq \frac{j-1}{J}\right\}$.

For a finite approximation, consider the market where $J-1$ sellers hold *second-price auctions* $\bar{\mu}_{-J} = \{\bar{\mu}_1, \ldots, \bar{\mu}_{J-1}\}$. The auctions generate cutoff valuations $\bar{y}_1$ through $\bar{y}_{J-1}$ and seller J, the deviating seller, holds some arbitrary incentive compatible direct mechanism μ'_J.[7] The distribution of the cutoff valuations $\bar{H}_J$ based on non-deviating sellers in the finite market converges almost everywhere to H. With a slight abuse of notation, let $\pi'_J(x, \mu'_J)$ be the probability with which the buyer with valuation x selects the deviating seller when his direct mechanism is μ'_J, given non-deviating sellers' auctions $\bar{\mu}_{-J}$. Lemma 2 in Peters (1997) shows that when a buyer's selection strategy is *incentive consistent* within non-deviating principals, a buyer with valuation $x \in [\bar{y}_{j-1}, \bar{y}_j)$ chooses each of the $j-1$ non-deviating sellers whose cutoff valuations are no greater than $\bar{y}_{j-1}$ with the same probability,

[6]Burguet and Sákovics (1999) show the existence of a mixed-strategy equilibrium in the two-seller case. Virag (2010) extends their result to any finite number of homogenous sellers and shows that the equilibrium reserve price converges to zero.

[7]The second price auction with the highest reserve price among all auctions in $\bar{\mu}_{-J}$ is denoted by $\bar{\mu}_{J-1}$. The second price auction with the second highest reserve price is then denoted by $\bar{\mu}_{J-2}$, and so on. Given this order of notation, the cutoff valuations have the following order: $\bar{y}_1 \leq \ldots \leq \bar{y}_{J-1}$.

$(1-\pi'_J(x,\mu'_J))/j-1$.[8] There are two implications of these results. First, for any given $\pi'_J(\cdot,\mu'_J)$, we can derive the incentive consistent strategies for selecting non-deviating sellers. Second, $\pi'_J(\cdot,\mu'_J)$ completely determines on its own the incentive consistent strategies for selecting non-deviating sellers. Therefore, unless specified otherwise, we will use π'_J to refer to the buyer's incentive consistent selection strategy.

Moving on, one can calculate the payoff $\bar{v}_1(x,\mu'_J,J)$ to a buyer with valuation x who selects the non-deviating seller offering the lowest reserve price:

$$\bar{v}_1(x,\mu'_J,J)=\int_{y_1}^{x}\left[1-\int_{\nu}^{1}\frac{1-\pi'_J(x,\mu'_J)}{n_J(s,H)}f(s)ds\right]^{\kappa J-1}d\nu, \tag{4}$$

where $n_J(x,H)=\max\{j:\bar{y}_j\leq x\}$ and $\frac{1-\pi'_J(x,\mu'_J)}{n_J(s,H)}$ is, according to Lemma 2 in Peters (1997), the probability with which a buyer with valuation x chooses the non-deviating seller with the lowest reserve price.[9] According to Peters (1997), the buyer must be indifferent on the margin between choosing the deviator and choosing any of the non-deviating auctioneers. Every non-deviating seller will provide the same expected payoff as the non-deviating seller who offers the lowest reserve price when the matching process is incentive consistent.

If we let $v'_J(x,\mu'_J,\pi'_J)$ denote the buyer's payoff upon selecting the deviating seller, we can characterize an incentive consistent continuation equilibrium upon a seller's deviation to any arbitrary incentive compatible direct mechanism as follows:

Remark 1 *Any incentive consistent continuation equilibrium upon a seller's deviation to an arbitrary incentive compatible direct mechanism is characterized by π'_J such that for every μ'_J and x,*

1. *π'_J induces incentive consistent strategies for selecting non-deviating sellers*

2. *$\pi'_J(x,\mu'_J)=0\Longrightarrow\bar{v}_1(x,\mu'_J,J)\geq v'_J(x,\mu'_J,\pi'_J)$.*

3. *$\pi'_J(x,\mu'_J)>0\Longrightarrow v'_J(x,\mu'_J,\pi'_J)\geq\bar{v}_1(x,\mu'_J,J)$.*

[8] If a buyer's valuation is no less than $\bar{y}_{J-1}$, she selects every non-deviating seller with equal probability. If a buyer's valuation is somewhere between $\bar{y}_{J-1}$ and $\bar{y}_{J-2}$, she selects, with equal probability, all non-deviating sellers except the seller who offers $\bar{\mu}_{J-1}$, and so on.

[9] All payoffs are conditional on the fixed distribution of cutoff valuations.

Note that $\bar{v}_1(x, \mu'_J, J)$ is the market payoff to buyers that is generated by the distribution of cutoff valuations H and the incentive consistent selection strategy π'_J.

2.3 A Competitive Distribution of Auctions

For any incentive consistent selection strategy π'_J, the deviating seller's payoff associated with offering an arbitrary incentive compatible direct mechanism is

$$\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = \\ w + \kappa J \int_0^1 \left[(x-w) Q_J(x, \mu'_J, \pi'_J) - \bar{v}_1(x, \mu'_J, J)\right] \pi'_J(x, \mu'_J) f(x) dx. \quad (5)$$

In addition, the payoff to the deviating seller with cost w who offers a second-price auction $\mu'(y)$ with cutoff valuation y is given by

$$\hat{\Phi}'_J(w, \mu'(y), H) = w + \\ \kappa J \int_0^1 \left[(x-w)\left[1 - \int_\nu^1 \frac{1}{n_J(s,H)+1} f(s) ds\right]^{\kappa J - 1} - \bar{v}_1(x, \mu'(y), J)\right] \times \\ \frac{1}{n_J(s,H)+1} f(x) dx.$$

Let $\Pi'_J(\mu'_J, H)$ be the set of all incentive consistent selection strategies π'_J that lead to incentive consistent continuation equilibrium given (μ'_J, H). Furthermore, let $\mathcal{M}^B_J(H)$ be the set of all incentive compatible direct mechanisms available for each seller's deviation given H. Then the definition of a competitive distribution of second-price auctions in Peters (1997) can be articulated as follows:

Definition 2 *A competitive distribution of second-price auctions is a distribution of cutoff valuations H and a cutoff rule $y : [0,1] \to [0,1]$ such that for almost all w,*

1. *for all $\mu'_J \in \mathcal{M}^B_J(H)$ and all $\pi'_J \in \Pi'_J(\mu'_J, H)$*

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \quad (6)$$

2. *$H(y(w)) = G(w)$.*

Theorem 5 in Peters (1997) shows that there is a competitive distribution of second-price auctions in which each seller offers a second-price auction with reserve price equal to his cost.[10] Let us explain how this result is established.

Peters (1997) shows that whichever direct mechanism the deviating seller picks, the market payoff $\bar{v}_1(x, \mu'(y), J)$ that any buyer gets in a finite approximation is bounded below by the market payoff $v^*(x, H)$ that she would get if the deviating seller were removed from the market entirely. That is to say, $\bar{v}_1(x, \mu'(y), J) \geq v^*(x, H)$.

Consider the situation where there are only $J - 1$ sellers in the market who offer second-price auctions with reserve prices that generate the cutoff valuations $\bar{H}_J$. Because there are no deviating sellers in this situation, the probability of trading with the seller who offers the lowest reserve price (i.e., seller 1) is given by

$$\left[1 - \int_{\nu}^{1} \frac{1}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1}. \tag{7}$$

It has already been established in subsection 2.2 that every non-deviating auctioneer will provide the same expected payoff as the non-deviating seller who offers the lowest reserve price when the matching process is incentive consistent. Therefore, the market payoff $v^*(x, H)$ is the same as the payoff that the buyer would experience by selecting seller 1. This can be derived by using (7) to create an expression that is analagous to (4).

Because $\bar{v}_1(x, \mu'(y), J) \geq v^*(x, H)$, the expected profit $\hat{\Phi}_J(w, \mu'_J, H, \pi'_J)$ of the deviating seller with a direct mechanism μ'_J satisfies

$$\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \leq \\ w + \kappa J \int_0^1 \left[(x - w) Q_J(s, \mu'_J, \pi'_J) - v^*(x, H))\right] \pi'_J(x, \mu'_J) f(x) dx \tag{8}$$

Following McAfee's (1993) approach, the market payoff $v^*(x, H)$ is taken as given. The value expressed on the right hand side of (8) can then be maximized over all possible incentive compatible direct mechanisms and all possible incentive consistent selection strategies.[11]

I shall now show the intuition of why the optimal direct mechanism is the (second-price) auction with reserve price equal to the seller's cost. If a deviating seller wants to attract a buyer, he needs to match her market payoff. It is shown that given this constraint, the optimal direct mechanism

[10] See Peters (1997) for how to retrieve the seller's reserve price from the cutoff valuation.

[11] See pages 115 and 116 in Peters (1997) for details.

must satisfy the upper bound of the implementability of the reduced-form mechanism (Border 1991). Because this upperbound is generated by an auction, the optimal direct mechanism for the deviating seller is an auction when everyone else offers an auction with reserve price equal to their cost. Since the deviating seller cannot make a profit from a buyer whose valuation is less than his cost, the reserve price is set equal to his cost.

Let Φ^*_J denote the maximum value of the left-hand side of (8). Peters (1997) shows that the limit payoff, $\lim_{J\to\infty}\Phi^*_J$, is in fact the same as the limit payoff, $\lim_{J\to\infty}\hat{\Phi}'_J(w,\mu'(y(w)),H)$, that accrues to the deviating seller when he offers an auction with reserve price w.

$$\lim_{J\to\infty}\Phi^*_J = \lim_{J\to\infty}\hat{\Phi}'_J(w,\mu'(y(w)),H) \tag{9}$$

On the other hand, because Φ^*_J is the maximum value of the left-hand side of (8), (8) implies that, for all $\mu'_J \in \mathcal{M}^B_J(H)$ and all $\pi'_J \in \Pi'_J(\mu'_J,H)$

$$\lim_{J\to\infty}\hat{\Phi}_J(w,\mu'_J,H,\pi'_J) \leq \lim_{J\to\infty}\Phi^*_J \tag{10}$$

Therefore, (9) and (10) yield (6) in the definition of a competitive distribution of second price auctions.

3 Robustness

In the literature on competing auctions, sellers may only offer auctions or direct mechanisms. However, a seller might want to offer an alternative mechanism if he can gain by doing so. We construct a continuation equilibrium upon a seller's deviation to an arbitrary mechanism, formulate the notion of robustness, and establish the robustness of competitive auctions.

3.1 Deviation to Arbitrary Mechanisms

Suppose that seller J deviates to a mechanism $\gamma_J = \{p^\gamma_J, q^\gamma_J\}$, where $p^\gamma_J : M^{\kappa J} \to \mathbb{R}$ specifies a buyer's payment and $q^\gamma_J : M^{\kappa J} \to [0,1]$ specifies the probability with which a buyer acquires the item being sold, with M being the set of messages that a buyer can send to the seller.[12] Let $M = M' \cup \{m^\circ\}$, where M' is the set of all messages that are available for a participating buyer

[12]Given that any direct mechanism is assumed to be anonymous in Peters (1997), any arbitrary mechanism is also assumed to be anonymous. However, the robustness result does not depend on the anonymity of a mechanism.

and m° is the message that is equivalent to "no participation." Suppose that $m \in M'$ is a buyer's message to seller J and that $\mathbf{m} \in M^{\kappa J-1}$ is the other buyers' message profile. Then $p_J^\gamma(m, \mathbf{m})$ denotes the price that the buyer pays to seller J and $q_J^\gamma(m, \mathbf{m})$ denotes the probability with which the buyer acquires the item.

After seller J's deviation to $\gamma_J = \{p_J^\gamma, q_J^\gamma\}$, given the other sellers' second-price auction offers, let $c'_J : X \to \Delta(M')$ be every buyer's communication strategy upon selecting J, with the exception of one buyer, say buyer number $J+1$, whose communication strategy is denoted by $c_J : X \to \Delta(M')$. Given communication strategies c_J and c'_J, one can induce a direct mechanism $\{\mathrm{p}_J^{c,c'}, \mathrm{q}_J^{c,c'}\}$. Let $N \leq \kappa J$ denote the number of participating buyers. Then the direct mechanism $\{\mathrm{p}_J^{c,c'}, \mathrm{q}_J^{c,c'}\}$ is induced as follows: For all $N \leq \kappa J$,

$$\mathrm{p}_J^{c,c'}(\mathbf{x}_N, \mathbf{x}_{-N}^\circ) = \int_{M'} \cdots \int_{M'} p_J^\gamma(\mathbf{m}_N, \mathbf{m}_{-N}^\circ) dc_J(x_1) \times dc'_J(x_2) \times \cdots \times dc'_J(x_N), \quad (11)$$

$$\mathrm{q}_J^{c,c'}(\mathbf{x}_N, \mathbf{x}_{-N}^\circ) = \int_{M'} \cdots \int_{M'} q_J^\gamma(\mathbf{m}_N, \mathbf{m}_{-N}^\circ) dc_J(x_1) \times dc'_J(x_2) \times \cdots \times dc'_J(x_N), \quad (12)$$

where $\mathbf{x}_N = (x_1, \ldots, x_N)$ is the profile of N participating agents' valuations, $\mathbf{m}_N = (m_1, \ldots m_N)$ is the profile of messages sent by participating agents, and $\mathbf{x}_{-N}^\circ$ and $\mathbf{m}_{-N}^\circ$ are the profiles of the messages that are equivalent to "no participation." Let $\acute{\pi}_J^\gamma(x, \gamma_J) \in [0, 1]$ denote the probability with which a buyer with valuation x selects seller J upon his deviation to γ_J. Then, with a slight abuse of notation, one can derive $z_J(\gamma_J, \acute{\pi}_J^\gamma)$, which is an analog of (1), and the reduced-form mechanism $\{Q_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma), P_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma)\}$, which is an analog of (2). Given that the other buyers select and communicate with seller J according to $\acute{\pi}_J^\gamma$ and c'_J, the payoff to a buyer with valuation x who sends a message to seller J according to the communication strategy c_J is

$$v_J(x, \gamma_J, c_J, c'_J, \acute{\pi}_J^\gamma) = \mathrm{Q}_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma)x - \mathrm{P}_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma).$$

If the buyer's communication strategy c_J is the same as c'_J that everyone else uses, then the direct mechanism is denoted by $\beta^{c'}(\gamma_J) = \{p_J^{c'}, q_J^{c'}\} = \{\mathrm{p}_J^{c',c'}, \mathrm{q}_J^{c',c'}\}$, the reduced-form mechanism by

$$\{Q_J^{c'}(x, \gamma_J, \acute{\pi}_J^\gamma), P_J^{c'}(x, \gamma_J, \acute{\pi}_J^\gamma)\} = \{\mathrm{Q}_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma), \mathrm{P}_J^{c,c'}(x, \gamma_J, \acute{\pi}_J^\gamma)\},$$

and the payoff to the buyer with valuation x by

$$v'_J(x, \gamma_J, c'_J, \acute{\pi}^\gamma_J) = v_J(x, \gamma_J, c'_J, c'_J, \acute{\pi}^\gamma_J).$$

We are interested in the incentive consistency of the buyer's strategy. Given Lemma 2 in Peters (1997), a buyer's incentive consistent selection probabilities across all non-deviating sellers are fully determined by the probability with which a buyer selects the deviating seller J, which is specified by $\acute{\pi}^\gamma_J$. Therefore, we will use $(c'_J, \acute{\pi}^\gamma_J)$ when we refer to the buyer's incentive consistent strategy. Let us define a buyer's payoff associated with selecting a seller with the lowest reserve price as follows:

$$\bar{v}_1(x, \gamma_J, J) = \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \acute{\pi}^\gamma_J(x, \gamma_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu,$$

where $\frac{1-\acute{\pi}^\gamma_J(x,\gamma_J)}{n_J(s,H)}$ is, according to Lemma 2 in Peters (1997), the probability with which a buyer with valuation x chooses the non-deviating seller with the lowest reserve price, given the buyer's incentive consistent selection strategy. Much like Remark 1, we can then characterize an incentive consistent continuation equilibrium upon seller J's deviation to an arbitrary mechanism.

Definition 3 $(c'_J, \acute{\pi}^\gamma_J)$ *is the buyer's incentive consistent strategy that characterizes an incentive consistent continuation equilibrium upon seller J's deviation to an arbitrary mechanism γ_J, that is,*

1. $\acute{\pi}^\gamma_J$ *induces incentive consistent strategies for selecting non-deviating sellers.*

2. $(c'_J, \acute{\pi}^\gamma_J)$ *satisfies the following conditions for the deviating seller: For all x*

   (a) $v'_J(x, \gamma_J, c'_J, \acute{\pi}^\gamma_J) \geq v_J(x, \gamma_J, c_J, c'_J, \acute{\pi}^\gamma_J)$ *for all c_J.*

   (b) $\acute{\pi}^\gamma_J(x, \gamma_J) = 0 \Longrightarrow \bar{v}_1(x, \gamma_J, J) \geq v'_J(x, \gamma_J, c'_J, \acute{\pi}^\gamma_J)$.

   (c) $\acute{\pi}^\gamma_J(x, \gamma_J) > 0 \Longrightarrow v'_J(x, \gamma_J, c'_J, \acute{\pi}^\gamma_J) \geq \bar{v}_1(x, \gamma_J, J)$.

Given the buyer's payoff $v_J(x, \gamma_J, c'_J, \acute{\pi}^\gamma_J)$, we can express seller J's payoff associated with offering an arbitrary mechanism γ_J as follows:

$$\Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J) = \kappa J \int_0^1 \left[(x - w) Q^{c'}_j(x, \gamma_J, \acute{\pi}^\gamma_J) - \bar{v}_1(x, \gamma_J, J)\right] \acute{\pi}^\gamma_J(x, \gamma_J) f(x) dx. \quad (13)$$

We define the robustness of a competitive distribution of second-price auctions in the next definition. Let Γ be a set of arbitrary mechanisms that a seller may consider for his deviation.

Definition 4 *A competitive distribution of second-price auctions is said to be robust to a set of mechanisms* Γ *if, for all* $(\acute{\pi}^{\gamma}_J, c'_J)$ *that characterize an incentive consistent continuation equilibrium upon a seller's deviation to any mechanism in* Γ*, the following condition is satisfied: For every* $\gamma_J \in \Gamma$*,*

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J).$$

3.2 Robust Competitive Auctions

Lemma 1 below is instrumental in establishing the robustness of competitive second-price auctions relative to any set of mechanisms Γ.

Lemma 1 *For any incentive consistent strategy* $(c'_J, \acute{\pi}^{\gamma}_J)$ *upon seller* J*'s deviation to an arbitrary mechanism* $\gamma_J \in \Gamma$*, there exist an incentive compatible direct mechanism* μ'_J *and an incentive consistent selection strategy* π'_J *such that*

$$\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J). \tag{14}$$

Proof. Let us take the buyer's communication strategy c'_J from her incentive consistent strategy $(c'_J, \acute{\pi}^{\gamma}_J)$, which characterizes a continuation equilibrium upon seller J's deviation to γ_J. By using c'_J, one can construct a direct mechanism $\beta^{c'}(\gamma_J) = \{p^{c'}_J, q^{c'}_J\}$ according to (11) and (12). Suppose that seller J offers a direct mechanism

$$\mu'_J = \beta^{c'}(\gamma_J)$$

instead of γ_J. Assume that the buyer selects seller J according to

$$\pi'_J(x, \mu'_J) = \acute{\pi}^{\gamma}_J(x, \gamma_J) \tag{15}$$

for all x. After seller J's deviation to μ'_J, a buyer selects each non-deviating seller with the same probability as would be the case if seller J deviated to γ_J. This preserves the incentive consistency of her strategy for selecting non-deviating sellers given the incentive compatibility of each non-deviating seller's second-price auction. Because buyers' selection probabilities are preserved, a buyer's payoff upon selecting any non-deviating seller is also preserved.

For the deviating seller J, it is worth noting that (15) means that the probability distribution z_J induced by π'_J given $\mu'_J = \beta^{c'}(\gamma_J)$ (the derivation in this case is similar to (1)) is the same as z_J induced $\acute{\pi}^{\gamma}_J$. Because the reduced-form mechanisms are based on z_J, this implies that the reduced-form mechanism from γ_J is the same as the reduced-form mechanism from $\mu'_J = \beta^{c'}(\gamma_J)$, given the construction of $\beta^{c'}(\gamma_J) = \{p^{c'}_J, q^{c'}_J\}$ according to (11) and (12):

$$\begin{aligned} Q_J(x, \mu'_J, \pi'_J) &= Q^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J), \\ P_J(x, \mu'_J, \pi'_J) &= P^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J). \end{aligned}$$

Because the reduced-form mechanisms are identical, the payoff to a buyer with valuation x who selects seller J is preserved by truthful valuation reporting;

$$Q_J(x, \mu'_J, \pi'_J)x - P_J(x, \mu'_J, \pi'_J) = Q^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J)x - P^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J). \quad (16)$$

If a buyer with valuation x reports x' to seller J who has offered μ'_J, her payoff is

$$Q_J(x', \mu'_J, \pi'_J)x - P_J(x', \mu'_J, \pi'_J) = Q^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J), \quad (17)$$

where the right-hand side is the buyer's payoff when she draws from the probability distribution $c'_J(x')$ her message to seller J who offers γ_J.

Given seller J's mechanism γ_J, suppose that a buyer with valuation x chooses from the probability distribution $c'_J(x')$, rather than $c'_J(x)$, a message to seller J. Then the buyer's payoff is $Q^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)$. Because $c'_J(x)$ is optimal for a buyer with valuation x in continuation equilibrium upon seller J's deviation to γ_J, we have

$$Q^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J)x - P^{c'}_J(x, \gamma_J, \acute{\pi}^{\gamma}_J) \geq Q^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^{c'}_J(x', \gamma_J, \acute{\pi}^{\gamma}_J). \quad (18)$$

(16) - (18) yield, for all $x, x' \in [0, 1]$,

$$Q_J(x, \mu'_J, \pi'_J)x - P_J(x, \mu'_J, \pi'_J) \geq Q_J(x', \mu'_J, \pi'_J)x - P_J(x', \mu'_J, \pi'_J)$$

so that $\mu'_J = \beta^{c'}(\gamma_J)$ is incentive compatible given the buyer's selection strategy π'_J. Because all sellers' mechanisms (including J's mechanism μ'_J) are incentive compatible given the buyer's same selection behavior, the buyer's payoff upon selecting any seller (including the one upon selecting μ'_J according to (16)) is preserved. Therefore, the incentive consistency of $\acute{\pi}^{\gamma}_J$ is preserved by π'_J.

Because the buyer's selection probability is preserved, a buyer's payoff associated with selecting a seller with the lowest reserve price is also preserved;

$$\begin{aligned}\bar{v}_1(x, \mu_J^{\gamma}, J) &= \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \pi'_J(x, \mu'_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu \\ &= \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \acute{\pi}_J^{\gamma}(x, \gamma_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu \\ &= \bar{v}_1(x, \gamma_J, J).\end{aligned} \tag{19}$$

(19) finally yields

$$\begin{gathered}\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = \\ \kappa J \int_0^1 \left[(x - w) Q_J(x, \mu'_J, \pi'_J) - \bar{v}_1(x, \mu'_J, J)\right] \pi'_J(x, \mu'_J) f(x) dx = \\ \kappa J \int_0^1 \left[(x - w) Q_j^{c'}(x, \gamma_J, \acute{\pi}_J^{\gamma}) - \bar{v}_1(x, \gamma_J, J)\right] \acute{\pi}_J^{\gamma}(x, \gamma_J) f(x) dx = \\ \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}_J^{\gamma}).\end{gathered} \tag{20}$$

Therefore, the deviating seller J's payoff is preserved. ■

Given Lemma 1, we establish the robustness of a competitive distribution of second-price auctions in Proposition 1 below.

Proposition 1 *A competitive distribution of second-price auctions is robust to any seller's deviation to any mechanism in any* Γ.

Proof. For any $\gamma_J \in \Gamma$, let $\mu'_J = \beta^{c'}(\gamma_J)$ be an incentive compatible direct mechanism that is induced by an incentive consistent strategy $(c'_J, \acute{\pi}_J^{\gamma})$, where $\beta^{c'}(\gamma_J) = \{p_J^{c'}, q_J^{c'}\}$ is constructed according to (11) and (12). Let $\Pi'_J(\mu'_J, H)$ be the set of all incentive consistent selection strategies π'_J given (μ'_J, H). (14) in Lemma 1 implies that, for any given $\gamma_J \in \Gamma_J$,

$$\sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}_J^{\gamma}). \tag{21}$$

Because μ'_J is incentive compatible (i.e., $\mu'_J \in \mathcal{M}_J^B(H)$), we have

$$\sup_{\mu'_J \in \mathcal{M}_J^B(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J). \tag{22}$$

(21) and (22) imply that for every $\gamma_J \in \Gamma$,

$$\lim_{J\to\infty} \sup_{\mu'_J \in \mathcal{M}^B_J(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J). \quad (23)$$

The first part of the definition of a competitive distribution of auctions (i.e., Part 1 of Definition 2) implies that for every $\gamma_J \in \Gamma$,

$$\lim_{J\to\infty} \Phi'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \sup_{\mu'_J \in \mathcal{M}^B_J(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \Phi_J(w, \mu'_J, H, \pi'_J). \quad (24)$$

Finally, from (23) and (24), we can conclude that for every $\gamma_J \in \Gamma$ and every incentive consistent strategy $(c'_J, \acute{\pi}^{\gamma}_J)$,

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J),$$

which implies that the seller's payoff upon deviation to any arbitrary mechanism γ_J is not greater than his payoff associated with the second-price auction with reserve price equal to his cost. Therefore, a competitive distribution of auctions is robust to the possibility that a seller may deviate to any arbitrary mechanism. ∎

Generally speaking, the mechanism offered by the deviating seller influences the buyer's communication with non-deviating sellers which, in turn, affects the buyer's selection probabilities across all sellers. The tractable feature of second-price auctions offered by non-deviating sellers is that we can always fix the buyer's communication with non-deviating sellers as truthful valuation reporting, regardless of the deviating seller's mechanism. This is because second-price auctions are dominant strategy incentive compatible so that the incentive to report the true valuation depends on neither a buyer's selection probabilities nor the mechanism offered by the deviating sellers. This makes it convenient to examine the robustness of the competitive second-price auctions by focusing on a seller's deviation to incentive compatible direct mechanisms given the buyer's truthful valuation reporting to any non-deviating seller upon selecting him.

When seller J deviates to an arbitrary mechanism γ_J, the buyer's communication strategy induces an incentive compatible direct mechanism from γ_J. Suppose that the deviation to γ_J induces an incentive compatible direct mechanism μ'_J and an incentive consistent strategy $(c'_J, \acute{\pi}^{\gamma}_J)$ for a buyer, whereas the deviation to a different mechanism γ'_J also induces μ'_J along with another incentive consistent strategy $(\tilde{c}_J, \tilde{\pi}^{\gamma'}_J)$. If there are multiple incentive consistent selection strategies upon J's deviation to μ'_J, then $\tilde{\pi}^{\gamma'}_J(\cdot, \gamma'_J)$ may be different from $\acute{\pi}^{\gamma}_J(\cdot, \gamma_J)$. In this case, the seller's payoff upon deviating to γ_J may be different from that upon deviating to γ'_J, even though the two different mechanisms induce the same incentive compatible direct mechanism μ'_J.

If seller J deviates to μ'_J, only one incentive consistent selection strategy π'_J is played and it may not induce the highest payoff of all the incentive consistent selection strategies upon such a deviation. Hence, a seller may want to deviate to a mechanism that is not available in the set of incentive compatible direct mechanisms if he expects an incentive consistent selection strategy that induces a higher payoff to the deviating seller. Therefore, in order to ensure the robustness of a distribution of competitive auctions, the payoff associated with each seller's auction should be no less than his payoff associated with *every* incentive compatible direct mechanism with *every* incentive consistent selection strategy, given the other sellers' auction choices. This coincides precisely with part 1 of Peters' (1997) definition of a competitive distribution of auctions, and it serves to explain why the competitive distribution of auctions is robust.

4 Markets with Complex Mechanisms

A seller may want to offer a more complex mechanism than an auction because he would like to make his terms of trade responsive to what happens with other competing sellers. This can be done by asking participating buyers to report not only their valuations but also market information (i.e., mechanisms offered by other competing sellers, their terms of trade, etc.). Mechanisms that allow this type of market information transmission should be equipped with a message space that is larger than the set of the buyers' valuations. To develop a clearer understanding of competition among sellers with arbitrarily complex mechanisms, we first formulate sellers' competition in a market with an arbitrary set of mechanisms.

Let $\bar{\Gamma}$ denote the set of mechanisms that each seller is allowed to offer in a market. A mechanism in $\bar{\Gamma}$ is denoted by $\gamma_j = \{p^{\gamma}_j, q^{\gamma}_j\}$, where $p^{\gamma}_j : \bar{M}^{\kappa J} \to \mathbb{R}$ specifies a buyer's payment and $q^{\gamma}_j : \bar{M}^{\kappa J} \to [0, 1]$ specifies the probability with which she acquires the item being sold. Let $\bar{M} = \bar{M}' \cup \{\bar{m}^{\circ}\}$, where $\bar{M}'$ is the set of all messages that are available for a participating buyer and $\bar{m}^{\circ}$ is equivalent to "no participation."

A mapping $c : X \times \bar{\Gamma}^J \to \Delta(\bar{M}')$ describes a buyer's communication behavior. A buyer with valuation x sends to seller j a message that is drawn from a probability distribution $c(x,\gamma_j,\gamma_{-j})\in\Delta(\bar{M}')$ when the seller's mechanism is γ_j and (the distribution of) the other sellers' mechanisms is γ_{-j}. A mapping $\pi : X\times\bar{\Gamma}^J\to[0,1]$ describes the probability that a buyer selects principal j. Thus, a buyer with valuation x selects seller j with probability $\pi(x_i,\gamma_j,\gamma_{-j})$.

Given $\gamma=(\gamma_j,\gamma_{-j})$, c induces the direct mechanism $\beta^c(\gamma_j)=\{p_j^c,q_j^c\}$ from γ_j, where p_j^c and q_j^c are derived in the same fashion as (11) and (12). The buyer's selection strategy π induces $z_j(\gamma_j,\pi)(x)$ as follows;

$$z_j(\gamma_j,\pi)(x)=1-\int_x^1\pi(s,\gamma_j,\gamma_{-j})f(s)ds.$$

$z_j(\gamma_j,\pi)(x)$ is the probability that a buyer's valuation is less than x or that she selects a seller other than j.

Suppose that a seller's strategy is $s:[0,1]\to\bar{\Gamma}$ and hence, a seller with cost w offers a mechanism $s(w)\in\bar{\Gamma}$. For a given (s,c), let $\bar{H}_{s,c}$ denote the distribution of direct mechanisms $\beta^c(s(w_k))\}_{k=1}^{J-1}$ that are induced by all sellers' but J's mechanisms when these sellers' costs are $(w_1,\dots,w_{J-1})$. Let $H_{s,c}$ be the distribution of direct mechanisms to which $\bar{H}_{s,c}$ converges as $J\to\infty$.

Note once again that the communication strategy c induces the direct mechanism $\beta^c(\gamma_j)=\{p_j^c,q_j^c\}$ from γ_j. Thus, the selection strategy π induces the reduced-form (direct) mechanism $\{Q_J^c(x,s(w),H_{s,c},\pi),P_J^c(x,s(w),H_{s,c},\pi)\}$ from $\beta^c(s(w))$ through $z_j(s(w),\pi)$. This is the reduced-form (direct) mechanism of $s(w)$ and it can be derived in the same way as (2).[13] Given (s,c,π), we can express in terms of the direct mechanism $\beta^c(s(w))$ the payoff to the buyer with valuation x who selects the seller offering the mechanism $s(w)$.

$$\hat{v}_J(x,\beta^c(s(w)),H_{s,c},\pi)=Q_J^c(x,s(w),H_{s,c},\pi)x-P_J^c(x,s(w),H_{s,c},\pi).$$

Let (c,π) denote an *incentive consistent continuation equilibrium* in a market with $\bar{\Gamma}$. It is optimal for a buyer to follow c and π for communication with and selection of a seller, respectively, when the other buyers do the same.

Given an incentive consistent continuation equilibrium (c,π), the profit to a seller with cost w, associated with mechanism $s(w)$, is

$$\begin{aligned}&\lim_{J\to\infty}\Phi_J(w,s(w),H_{s,c},c,\pi)=w+\\&\lim_{J\to\infty}\left[\int_0^1\left[(x-w)Q_J^c(x,s(w),H_{s,c},\pi)-\hat{v}_J(x,\beta_J^c(s(w)),H_{s,c},\pi)\right]dz_j(s(w),\pi)\right].\end{aligned}\tag{25}$$

[13] $H_{s,c}$ in the reduced-form mechanism implies that the reduced form is derived based on the distribution $\bar{H}_{s,c}$ of direct mechanisms that converges to $H_{s,c}$ as $J\to\infty$

We are interested in whether the distribution of direct mechanisms $H_{s,c}$ induced by (s, c) can be supported as an equilibrium (s, c, π) that is robust to the possibility that a seller may deviate to a mechanism in a set of mechanisms Γ that may not be available in the market with $\bar{\Gamma}$.

Suppose that seller J deviates to an arbitrary mechanism $\gamma_J = \{p_J^\gamma, q_J^\gamma\} \in \Gamma$, with $p_J^\gamma : M^{\kappa J} \to \mathbb{R}$ and $q_J^\gamma : M^{\kappa J} \to [0,1]$, where $M = M' \cup \{m^\circ\}$. Let $c' : X \times \Gamma \times \bar{\Gamma}^{J-1} \to \Delta(M')$ be a buyer's communication strategy for deviating seller J. The probability distribution that a buyer with valuation x uses for her communication with deviating principal J at (γ_J, γ_{-J}) is therefore denoted by $c'(x, \gamma_J, \gamma_{-J})$. In addition, if we let $\pi' : X \times \Gamma \times \bar{\Gamma}^{J-1} \to [0,1]$ represent a buyer's strategy for selecting deviating seller J, the probability with which a buyer with valuation x selects deviating seller J at (γ_J, γ_{-J}) is denoted by $\pi'(x, \gamma_J, \gamma_{-J})$.

Let (c', π') be a profile of a buyer's communication and selection strategies for deviating seller J. We can similarly define a profile of communication and selection strategies for non-deviating sellers, (c°, π°). Let a profile of the buyer's strategies $(c^\circ, \pi^\circ, c', \pi')$ be incentive consistent, that is, let it constitute an incentive consistent continuation equilibrium upon a seller's deviation to a mechanism in Γ. Given c', one can derive the direct mechanism $\beta^{c'}(\gamma_J)$ that γ_J induces. Given c°, H_{s,c° denotes the distribution of direct mechanisms that non-deviating sellers' mechanisms induce in the limit. The deviating seller J's profit $\lim_{J\to\infty} \Phi_J(w, \gamma_J, H_{s,c^\circ}, c', \pi')$ can then be derived in the same fashion as (25).

4.1 Market Information-Free Robust Equilibrium

Many equilibrium allocations in a market with complex mechanisms hinge on buyers' reports on whether there is a competing seller who has deviated from their implicit agreement (regarding prices, for instance). Yamashita (2010) shows how market information transmission can be understood; in the recommendation mechanism he proposes, a seller can ask buyers to report the direct mechanism that he should implement, in addition to their valuation. If there are three or more participating buyers and the majority of them recommend the same direct mechanism, then it is implemented by the seller. If a competing seller deviates, buyers, upon selecting a non-deviator, all recommend a direct mechanism that can punish the deviator. Recommending to a non-deviator that he should implement the direct mechanism μ' has the same effect as the non-deviator changing his direct mechanism to μ' upon buyers' reports on the deviator's mechanism.[14]

However, equilibria that support outcomes through market information transmission may be vulnerable to buyers' collusion. Suppose that an equilibrium can only be supported when buyers recommend the "right" direct mechanism, that is, the one that can successfully punish the deviator. In other words, a seller has incentives to deviate if non-deviators do not change their direct mechanisms upon his deviation. This type of equilibrium will disappear if, upon a seller's deviation, buyers all continue to recommend to a non-deviator the direct mechanism that he is supposed to implement on the path with no seller's deviation.[15]

We would therefore like to propose a notion of equilibrium in a market with $\bar{\Gamma}$ that is not only robust to a seller's deviation to any arbitrary mechanism but also independent of market information transmission from buyers to sellers. Let us define market information-free robust equilibrium (s, c, π). Let Λ be the set of all incentive consistent strategy profiles upon a seller's deviation. Let $\mathcal{C}^\circ$ be the projection of Λ onto the space of strategies for communicating with a non-deviating seller. For any $c^\circ \in \mathcal{C}^\circ$, let $\Lambda'(c^\circ) = \{(c', \pi') : \exists \pi' \text{ s.t. } (c^\circ, \pi^\circ, c', \pi') \in \Lambda\}$.

Definition 5 *An equilibrium (s, c, π) in a market with $\bar{\Gamma}$ is a market information-free robust equilibrium if there exists $c^\circ \in \mathcal{C}^\circ$ upon a seller's deviation to a mechanism in any Γ such that*

1. market information-free*: for all $\gamma_J \in \Gamma$, all $(w_1, \ldots, w_J) \in [0,1]^J$, and all $x \in [0,1]$*

$$c^\circ(x, s(w_\ell), (\gamma_J, \{s(w_k)\}_{k \neq \ell, J})) = c(x, s(w_\ell), \{s(w_k)\}_{k \neq \ell}),$$

where $(\gamma_J, \{s(w_k)\}_{k \neq \ell, J})$ and $\{s(w_k)\}_{k \neq \ell}$ on both sides are the distribution of sellers' mechanisms, with the exception of seller ℓ, and

[14] For example, in the recommendation mechanism, buyers all recommend an auction with reserve price equal to higher than the seller's cost on the path with no seller's deviation. Off the path following some seller's deviation, buyers all recommend to a non-deviator an auction with reserve price equal to zero. This type of communication behavior in the recommendation mechanism may support auctions with high reserve prices because lowering the reserve price down to zero upon a competing seller's deviation will dampen the seller's incentive to deviate in the first place.

[15] Buyers' collusion in the reporting of market information can even arise when no seller deviates. Consider an equilibrium where buyers are supposed to recommend to every seller an auction with reserve price higher than cost when no seller deviates. Buyers can all collude to recommend an auction with reserve price equal to the seller's cost (or even equal to zero), even without any seller's deviation. The equilibrium will disappear in this case as well.

2. robustness*: for all* $(c', \pi') \in \Lambda'(c^\circ)$ *and all* $\gamma_J \in \Gamma$

$$\lim_{J\to\infty} \Phi_J(w, s(w), H_{s,c}, c, \pi) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H_{s,c^\circ}, c', \pi'). \tag{26}$$

Condition 1 in Definition 5 is the essence of the market information-free property. Even when a seller deviates to γ_J, the buyer sends to a non-deviating seller a message drawn from the same probability distribution as the one she uses when sending a message to each seller on the path without any seller's deviation. For example, in Yamashita's recommendation mechanism, together with truthful valuation reporting, the buyer always recommends to every seller the direct mechanism that each seller is supposed to implement on the path irrespective of whether there is a deviating seller. This is why we call it "market information-free." Importantly, this induces $H_{s,c} = H_{s,,c^\circ}$. This implies that when seller J considers deviation, he takes the (market) distribution of the other sellers' direct mechanisms as given.

Condition 2 for robustness was defined earlier in Definition 4. It ensures that no seller has incentives to deviate to any arbitrary mechanism, let alone a mechanism that is available in the market.

4.2 Market Information-Free Robust Allocation

Let us consider an allocation in the economy. An allocation is characterized by sellers' (incentive compatible) direct mechanisms and buyers' selection decisions. Suppose that the direct mechanism chosen by a seller with cost w is characterized by a mapping $\mu^* : [0,1] \to \Omega$, where Ω is the set of all possible direct mechanisms. For any $w \in [0,1]$, let $\Omega^*(w) = \{\mu^*(w') \in \Omega \; \forall w' \in [0,w]\}$. Then the market distribution of direct mechanisms associated with μ^* is given by H^*, such that $H^*(\Omega^*(w)) = G(w)$. Defined Ω^* as

$$\Omega^* = \{\mu^*(w) \in \Omega \; \forall w \in [0,1]\} \tag{27}$$

Let the buyer's selection decision be characterized by a mapping $\pi^* : [0,1] \times \Omega^J \to [0,1]$, so that $\pi^*(x, \mu_j, \mu_{-j})$ is the probability with which the buyer selects the seller who offers a direct mechanism μ_j when (the distribution of) the other sellers' direct mechanisms is given by μ_{-j}. Let (H^*, π^*) be an allocation where each direct mechanism in H^* is incentive compatible with π^*.

We now provide a characterization of an allocation (H^*, π^*) that can be supported by a market information-free robust equilibrium in a market with any set of mechanisms. Following the same notation as before, let $\Pi'_J(\mu'_J, H^*)$

be the set of all incentive consistent strategies for selecting the deviating seller in continuation equilibrium upon a seller's deviation to $\mu'_J \in \mathcal{M}_J^B(H^*)$ given H^*, where $\mathcal{M}_J^B(H^*)$ is the set of all incentive compatible direct mechanisms available for each seller's deviation given H^*.

Theorem 1 *Suppose that an allocation (H^*, π^*) is supported by a market information-free robust equilibrium (s, c, π) in a market with $\bar{\Gamma}$. It can be supported by a market information-free robust equilibrium (μ^*, π^*) in a market with $\Omega^* \subset \Omega$, that is,*

1. market information-free*: A buyer reports her true valuation to each non-deviating seller upon participation.*

2. robustness*:*

$$\lim_{J\to\infty} \hat{\Phi}_J(w, \mu^*(w), H^*, \pi^*) \geq \lim_{J\to\infty} \sup_{\mu'_J \in \mathcal{M}_J^B(H^*)} \sup_{\pi' \in \Pi'_J(\mu'_J, H^*)} \hat{\Phi}_J(w, \mu'_J, H^*, \pi') \tag{28}$$

Proof. Suppose that (H^*, π^*) is supported by a market information-free robust equilibrium (s, c, π) in a market with $\bar{\Gamma}$. The corresponding equilibrium allocation (H^*, π^*) is derived according to

$$H^* = H_{s,c} = H_{s,c^\circ} \tag{29}$$

$$\pi^*(x, \beta^c(s(w_j)), \beta^c(s(w_{-j}))) = \pi(x, s(w_j), s(w_{-j})), \tag{30}$$

where $\beta^c(s(w_{-j})) = \{\beta^c(s(w_k))\}_{k\neq j}$ and $s(w_{-j}) = \{s(w_k)\}_{k\neq j}$. Note that $H_{s,c} = H_{s,c^\circ}$ in (29) comes from the "market information-free" condition in Definition 5.

Suppose that the market now allows only the set of direct mechanisms $\{\beta^c(s(w)) \in \Omega \ \forall w \in [0,1]\}$. Let a seller's strategy in this market be $\mu^* : [0,1] \to \Omega^*$ with

$$\mu^*(w) = \beta^c(s(w)) \tag{31}$$

for all $w \in [0,1]$. Given μ^* in (31), the set of direct mechanisms $\{\beta^c(s(w)) \in \Omega \ \forall w \in [0,1]\}$ that is allowed in the market is the same as Ω^* defined in (27).

Note that the "market information-free" property of the equilibrium (s, c, π) that was specified in Definition 5 implies that the direct mechanism that a non-deviating seller j's equilibrium mechanism induces given c° is the same as that which it induces given c, regardless of a deviating seller's mechanism. That is, $\beta^{c^\circ}(s(w_j)) = \beta^c(s(w_j))$. Furthermore, because c° and c are part of the continuation equilibrium on and off the path, $\beta^{c^\circ}(s(w_j)) = \beta^c(s(w_j))$ is incentive compatible regardless of whether there is a deviating seller. Therefore, we can fix truthful valuation reporting as the buyer's strategy for communicating with a non-deviating seller. If sellers offer mechanisms in Ω^*, the buyer's selection strategy is π^* constructed according to (30). It is then straightforward to show that (μ^*, π^*) is an equilibrium in the market with Ω^* given equilibrium properties of (s, c, π). Moreover, the fact that we fix truthful valuation reporting to every non-deviating seller shows that the "market information-free" condition of (μ^*, π^*) is satisfied.

For the robustness of (μ^*, π^*), note that in the market with Ω^*, the profit for the seller with cost w is

$$\lim_{J\to\infty} \hat{\Phi}_J(w, \mu^*(w), H^*, \pi^*) = \lim_{J\to\infty} \Phi_J(w, s(w), H_{s,c}, c, \pi) \tag{32}$$

because of (29) - (31). We can extend Lemma 1 to show that for any $(c', \pi') \in \Lambda'(c^\circ)$ and all $\gamma_J \in \Gamma$, there exists an incentive compatible direct mechanism (i.e., $\mu'_J = \beta^{c'}(\gamma_J)$) and an incentive consistent selection strategy $\tilde{\pi}'$ such that

$$\hat{\Phi}_J(w, \mu'_J, H_{s,c^\circ}, \tilde{\pi}') = \Phi_J(w, \gamma_J, H_{s,c^\circ}, c', \pi'). \tag{33}$$

Because $\mu'_J \in \mathcal{M}^B_J(H_{s,c^\circ})$ and $\tilde{\pi}' \in \Pi'_J(\mu'_J, H_{s,c^\circ})$, (33) implies that the robustness condition of (s, c, π) specified in Definition 5 is equivalent to the following condition:

$$\lim_{J\to\infty} \Phi_J(w, s(w), H_{s,c}, c, \pi) \geq \lim_{J\to\infty} \sup_{\mu'_J \in \mathcal{M}^B_J(H_{s,c^\circ})} \sup_{\pi' \in \Pi'_J(\mu'_J, H_{s,c^\circ})} \hat{\Phi}_J(w, \mu'_J, H_{s,c^\circ}, \pi'). \tag{34}$$

Because of (29) and (32), (34) is equivalent to (28).

Now the only thing we need to show is that the second condition expressed by (28) implies the robustness expressed by (26), that is, once an equilibrium (μ^*, π^*) is reached in the market with Ω^*, a deviating seller cannot gain in any incentive consistent continuation equilibrium upon his deviation to any arbitrary mechanism where buyers report their true valuation to non-deviating sellers.

In equilibrium (μ^*, π^*), the market distribution of direct mechanisms is H^*. Suppose that a seller deviates to a mechanism in Γ while the other sellers offer direct mechanisms according to μ^*. According to the first condition in the theorem, we fix buyers' truthful valuation reporting to non-deviating sellers upon participation. Then, let $\tilde{\Lambda}'$ be the set of all feasible buyer's incentive consistent selection and communication strategies vis-à-vis the deviating seller upon his deviation to a mechanism in any Γ, given the distribution of non-deviating sellers' equilibrium mechanisms H^*. By replacing H_{s,c° with H^*, (33) becomes as follows: For any $(c', \pi') \in \tilde{\Lambda}'$ and all $\gamma_J \in \Gamma$, there exists an incentive compatible direct mechanism (i.e., $\mu'_J = \beta^{c'}(\gamma_J)$) and an incentive consistent selection strategy $\tilde{\pi}'$ such that

$$\hat{\Phi}_J(w, \mu'_J, H^*, \tilde{\pi}') = \Phi_J(w, \gamma_J, H^*, c', \pi') \tag{35}$$

(35) implies that (28) is equivalent to the following statement: For any $(c', \pi') \in \tilde{\Lambda}'$ and all $\gamma_J \in \Gamma$,

$$\lim_{J\to\infty} \hat{\Phi}_J(w, \mu^*(w), H^*, \pi^*) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H^*, c', \pi')$$

This completes the proof. ■

Consider any allocation that is implemented by a "market information-free robust equilibrium" in a market with a set of mechanisms. The theorem above shows that it is also implemented by a market information-free robust equilibrium with only a subset of direct mechanisms in which (i) each seller's equilibrium direct mechanism is incentive compatible not only on the path but also off the path following a seller's deviation to any arbitrary mechanism in every incentive consistent continuation equilibrium, no seller can gain upon offering any possible incentive compatible direct mechanism. Condition 1 implies the "market information-free" property of the equilibrium in a market with only a subset of direct mechanisms and condition 2 ensures the "robustness" of the equilibrium.

Therefore, in the context of a market information-free robust equilibrium allocation, we can focus on direct mechanisms in terms of mechanisms that a market should allow. The market information-free property makes the analysis of robustness tractable. For robustness, we only need to consider a seller's deviation to incentive compatible direct mechanisms, but the incentive compatibility is endogenous because a buyer's payoff upon selecting the deviating seller depends on the probability with which other buyers select him. The probability that a buyer selects the deviator depends, in turn, on the distribution of non-deviating sellers' direct mechanisms that are induced by the buyer's communication with non-deviating sellers.[16] Because of the market information-free property, we can fix truthful valuation reporting to non-deviating sellers. Therefore, the distribution of non-deviating sellers' direct mechanisms is always the same, regardless of the deviating seller's mechanism.

[16]The introduction explains how this makes it difficult to specify the set of incentive compatible direct mechanisms for the deviating seller.

Given this distribution of non-deviating sellers' direct mechanisms, the set of all possible incentive compatible direct mechanisms includes any incentive compatible direct mechanism that can be induced in any continuation equilibrium upon the deviating seller's arbitrary mechanism. This is why we only need to consider the set of incentive compatible direct mechanisms given the distribution of non-deviating sellers' direct mechanisms. This is indeed how the robustness of a competitive distribution of auctions is established in Section 3. We used the fact that second-price auctions are *dominant strategy incentive compatible*; truthful valuation reporting to non-deviating sellers is fixed in continuation equilibrium whether a deviating seller offers an arbitrary mechanism or a direct mechanism. This makes it possible to fix the distribution of non-deviating sellers' direct mechanisms (i.e., second-price auctions), regardless of the deviating seller's mechanism.

5 Conclusion

Given the immense popularity of trading through online auctions, a lot of attention has been paid from both a theoretical and a practical viewpoint to how competing sellers choose to offer auctions. However, the difficulty is that optimal auction theory, as pioneered by Myerson (1981), is an inadequate analytical tool because it was developed for a monopolistic seller. The literature on competing auctions shows that no seller can gain by offering any direct mechanism instead of an auction with reserve price equal to his cost. However, it is not known whether a seller has incentives to deviate to any other selling mechanism. Therefore, despite the widespread adoption of auctions as selling mechanisms for competing sellers in practice, its optimality has only been explained using ad hoc restrictions on the set of direct mechanisms.

This paper provides an optimal auction theory for sellers in a competitive market. It shows that if the second-price auction with reserve price equal to cost is optimal for a seller among direct mechanisms, given that other competing sellers offer second-price auctions with reserve price equal to their cost, it is also optimal among all possible mechanisms. This global optimality of the second-price auction among all possible mechanisms is established by showing that the robustness of a competitive distribution of second-price auctions is embedded in the notion of equilibrium in Peters (1997).

The characterization of market information-free robust equilibrium provides a better understanding of a robust equilibrium allocation that does not depend on the transmission of market information from buyers to sellers. Because second-price auctions allocate the object to the participant with the highest valuation among all participants including the seller himself, the competitive distribution of second-price auctions induces a (constrained) ex-post efficient allocation through market information-free robust equilibrium in a market with a set of mechanisms as small as the set of second-price auctions with reserve prices.

References

[1] Border, K. C., (1991), " Implementation of reduced form auctions: A geometric approach," *Econometrica*, 59, 1175-1187

[2] Burguet, R. and J. Sákovics (1999), "Imperfect competition in auction designs." *International Economic Review*, 40, 231–247.

[3] Epstein, L., and M. Peters, (1999), "A revelation principle for competing mechanisms," *Journal of Economic Theory*, 88, 119–160.

[4] McAfee, P. (1993), "Mechanism design by competing sellers," *Econometrica*, 61, 1281-1312.

[5] Myerson, R. (1981), "Optimal auction design," *Mathematics of Operations Research*, 6, 58-73

[6] Peck, J., (1997), "A note on competing mechanisms and the revelation principle," Ohio State University, mimeo.

[7] Peters, M. (1997), "A competitive distribution of auctions," *Review of Economic Studies*, 64, 97-123.

[8] Peters, M. and Severinov, S. (1997), "Competition among sellers who offer auctions instead of prices," *Journal of Economic Theory*, 75, 141-179.

[9] Peters, M., and C. Troncoso-Valverde (2013), "A folk theorem for competing mechanisms," *Journal of Economic Theory*, 148(3), 953–973.

[10] Virág, G. (2010), "Competing auctions: Finite markets and convergence," *Theoretical Economics*, 2010, 5, 241-274.

[11] Yamashita, T. (2010), "Mechanism games with multiple principals and three or more agents," *Econometrica*, 2010, 78(2), 791-801.