MATHEMATICAL PSYCHICS

AN ESSAY ON THE

APPLICATION OF MATHEMATICS TO

THE MORAL SCIENCES

BY

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LONDON

C. KEGAN PAUL & CO., 1 PATERNOSTER SQUARE

1881

INTRODUCTORY

DESCRIPTION OF

CONTENTS.

MATHEMATICAL PSYCHICS may be divided into two parts—Theoretical and Applied.

In the First Part (1) it is attempted to illustrate the possibility of Mathematical reasoning without *numerical* data (pp. 1–7); without more precise data than are afforded by estimates of *quantity of pleasure* (pp. 7–9). (2) An analogy is suggested between the *Principles of Greatest Happiness*, Utilitarian or Egoistic, which constitute the first principles of Ethics and Economics, and those *Principles of Maximum Energy* which are among the highest generalisations of Physics, and in virtue of which mathematical reasoning is applicable to physical phenomena quite as complex as human life (pp. 9–15).

The Calculus of Pleasure (Part II.) may be divided into two species—the Economical and the Utilitarian; the principle of division suggesting an addition to Mr. Sidgwick's 'ethical methods' (p. 16).

The first species of *Calculus* (if so ambitious a title may for brevity be applied to short studies in Mathematical Economics) is developed from certain *Definitions* of leading conceptions, in particular of those connected with *Competition* (pp. 17–19). Then (α) a mathematical theory of *Contract unqualified by Competition* is given (pp. 20–30). (β) A mathematical theory of *Contract determined by Competition in a perfect Market* is given, or at least promised (pp. 30–33, and pp. 38–42). Reference is made to other mathematical theories of Market, and to Mr. Sidgwick's recent article on the 'Wages-Fund' (pp. 32, 33, and Appendix V.) (γ) attention is concentrated on the question—*What is a perfect Market?* It is argued that Market is imperfect, *Contract is indeterminate* in the following cases:—

(I.) When the number of competitors is limited (pp. 37, 39).

(II.) In a certain similar case likely to occur in contracts for *personal service* (pp. 42, 46).

(I. and II.) When the *articles* of contract are not perfectly divisible (p. 42, 46).

(III.) In case of *Combination*, Unionism; in which case it is submitted that (in general and abstractly speaking) *unionists stand to gain* in senses contradicted or ignored by distinguished economists (pp. 44, 47, 48).

(IV.) In a certain case similar to the last, and likely to occur in *Co-operative Association* (pp. 45, 49).

The *indeterminateness* likely from these causes to affect *Commercial Contracts*, and certainly affecting all sorts of *Political Contracts*, appears to postulate a *principle of arbitration* (pp. 50–52).

It is argued from mathematical considerations that *the basis of arbitration between contractors is the greatest possible utility of all concerned;* the Utilitarian first principle, which can of course afford only a general

direction—yet, as employed by Bentham's school, has afforded *some* direction in practical affairs (pp. 53–56).

The Economical thus leads up to the Utilitarian species of Hedonics; some studies in which already published [1] (under the title of 'Hedonical Calculus'—the species being designated by the generic title) are reprinted here by the kind permission of the Editor of 'Mind.'

Of the Utilitarian Calculus (pp. 56–82) the central conception is *Greatest Happiness*, the greatest possible sum-total of pleasure summed through all time and over all sentience. Mathematical reasonings are employed partly to confirm Mr. Sidgwick's proof that Greatest Happiness is the *end* of right action; partly to deduce middle axioms, *means* conducive to that end. This deduction is of a very abstract, perhaps only negative, character; negativing the assumption that *Equality* is necessarily implied in Utilitarianism. For, if sentients differ in *Capacity for happiness*—under similar circumstances some classes of sentients experiencing on an average more pleasure (*e.g.* of imagination and sympathy) and less pain (*e.g.* of fatigue) than others—there is no presumption that equality of circumstances is the most felicific arrangement; especially when account is taken of the interests of posterity.

Such are the principal topics handled in this *essay* or *tentative* study. Many of the topics, tersely treated in the main body of the work, are more fully illustrated in the course of seven supplementary chapters, or APPENDICES, entitled:

[1] *Mind*, July 1879.

Discussions too much broken up by this arrangement are re-united by references to the principal headings, in the Index; which also refers to the definitions of terms used in a technical sense. The Index also contains the names of many eminent men whose theories, bearing upon the subject, have been noticed in the course of these pages. Dissent has often been expressed. In so terse a composition it has not been possible always to express, what has always been felt, the deference due to the men and the diffidence proper to the subject.

MATHEMATICAL PSYCHICS.

ON THE APPLICATION OF MATHEMATICS TO THE MORAL SCIENCES.

The application of mathematics to *Belief*, the calculus of Probabilities, has been treated by many distinguished writers; the calculus of *Feeling*, of Pleasure and Pain, is the less familiar, but not in reality[1] more paradoxical subject of this essay.

The subject divides itself into two parts; concerned respectively with principle and practice, root and fruit, the applicability and the application of Mathematics to Sociology.

PART I.

In the first part it is attempted to prove an affinity between the moral and the admittedly mathematical sciences from their resemblance as to (1) a certain general complexion, (2) a particular salient feature.

(1) The science of quantity is not alien to the study of man, it will be generally admitted, in so far as actions and effective desires can be *numerically* measured by way of statistics—that is, very far, as Professor Jevons[2] anticipates. But in so far as our *data* may consist of

[1] Cf. Jevons, *Theory*, p. 9.
[2] Introduction to *Theory of Political Economy*.

estimates other than *numerical*, observations that some conditions are accompanied with *greater* or *less* pleasure than others, it is necessary to realise that mathematical reasoning is not, as commonly[1] supposed, limited to subjects where numerical data are attainable. Where there are data which, though not *numerical* are *quantitative*—for example, that a quantity is *greater* or *less* than another, *increases* or *decreases*, is *positive* or *negative*, a *maximum* or *minimum*, there mathematical reasoning is possible and may be indispensable. To take a trivial instance: *a* is greater than *b*, and *b* is greater than *c*, therefore *a* is greater than *c*. Here is mathematical reasoning applicable to quantities which may not be susceptible of numerical evaluation. The following instance is less trivial, analogous indeed to an important social problem. It is required to distribute a given quantity[2] of fuel, so as to obtain the greatest possible quantity of available energy, among a given set of engines, which differ in efficiency—*efficiency* being thus defined: one engine is more efficient than another if, whenever the total quantity of fuel consumed by the former is equal to that consumed by the latter, the total quantity of energy yielded by the former is greater than that yielded by the latter.

In the distribution, shall a larger portion of fuel be given to the more efficient engines? always, or only in some cases? and, if so, in what sort of cases? Here is a very simple problem involving no numerical data, yet

[1] The popular view pervades much of what Mill (in his *Logic*), after Comte, says about Mathematics applied to Sociology. There is a good expression of this view in the *Saturday Review* (on Professor Jevons's *Theory*, November 11, 1871.) The view adopted in these pages is expressed by Cournot, *Recherches*.)

[2] *Or*, a given quantity *per unit of time*, with corresponding modification of definition and problem.

requiring, it may be safely said, mathematics for its complete investigation.

The latter statement may be disputed in so far as such questions may be solved by reasoning, which, though not symbolical, is strictly mathematical; answered more informally, yet correctly, by undisciplined common sense. But, firstly, the advocate of mathematical reasoning in social science is not concerned to deny that mathematical reasoning in social, as well as in physical, science may be divested of symbol. Only it must be remembered that the question how far mathematics can with safety or propriety be divested of her peculiar costume is a very delicate question, only to be decided by the authority and in the presence of Mathematics herself. And, secondly, as to the sufficiency of common sense, the worst of such unsymbolic, at least unmethodic, calculations as we meet in popular economics is that they are apt to miss the characteristic advantages of deductive reasoning. He that will not verify his conclusions as far as possible by mathematics, as it were bringing the ingots of common sense to be assayed and coined at the mint of the sovereign science, will hardly realize the full value of what he holds, will want a measure of what it will be worth in however slightly altered circumstances, a means of conveying and making it current. When the given conditions are not sufficient to determinate the problem —a case of great importance in Political Economy— the ἀγεωμετρητὸς is less likely to suspect this deficiency, less competent to correct it by indicating what conditions are necessary and sufficient. All this is evident at a glance through the instrument of mathematics, but to the naked eye of common sense partially and obscurely, and, as Plato says of unscientific knowledge, in a state between genuine Being and Not-Being.

The preceding problem, to distribute a given quantity of material in order to a maximum of energy, with its starting point *loose quantitative relations* rather than numerical data—its slippery though short path almost necessitating the support of mathematics—illustrates fairly well the problem of utilitarian distribution.[1] To illustrate the economical problem of exchange, the maze of many dealers contracting and competing with each other, it is possible to imagine[2] a mechanism of many parts where the law of motion, which particular part moves off with which, is not precisely given—with symbols, arbitrary functions, representing not merely *not numerical knowledge* but[3] *ignorance*—where, though the mode of motion towards equilibrium is indeterminate, the position of equilibrium is mathematically determined.

Examples not made to order, taken from the common stock of mathematical physics, will of course not fit so exactly. But they may be found in abundance, it is submitted, illustrating the property under consideration—mathematical reasoning without numerical data. In Hydrodynamics, for instance, we have a Thomson or Tait[4] reasoning 'principles' for 'determining P and Q *will be given later.* In the meantime it is obvious that *each decreases as X increases.* Hence the equations of motion show'—and he goes on to draw a conclusion of

[1] See p. 64.
[2] See p. 34.
[3] *Ignoration of Co-ordinates* (Thomson and Tait, *Natural Philosophy*, 2nd edition), is appropriate in many social problems where we only know in part.
[4] Thomson and Tait, *Treatise on Natural Philosophy*, p. 320, 2nd edition. The italics, which are ours, call attention to the *unnumerical, loose quantitative, relation* which constitutes the datum of the mathematical reasoning.

momentous interest that balls (properly) projected in an infinite incompressible fluid will move as if they were attracted to each other. And generally in the higher Hydrodynamics, in that boundless ocean of perfect fluid, swum through by vortices, where the deep first principles of Physics are to be sought, is not a similar *unnumerical*, or *hyperarithmetical* method there pursued? If a portion of perfect fluid so moves at any time that each particle has no motion of rotation, then that portion of the fluid will retain that property for all time[1]; here is no application of the numerical measuring-rod.

No doubt it may be objected that these hydrodynamical problems employ some *precise* data; the very definition of Force, the conditions of fluidity and continuity. But so also have our social problems *some* precise data: for example, the property of *uniformity of price* in a market; or rather the (approximately realised) conditions of which that property is the deducible *effect*, and which bears a striking resemblance to the data of hydrodynamics:[2] (1) the *fulness* of the market: that there *continues* to be up to the conclusion of the dealing an indefinite number of dealers; (2) the *fluidity* of the market, or infinite dividedness of the dealers' interests. Given this property of uniform price, Mr. Marshall and M. Walras deduce mathematically, though not arithmetically, an interesting theorem, which Mill and Thornton failed with unaided reason to discern, though they were quite close to it—the theorem that the equation of supply to demand, though a necessary, is not a sufficient condition of market price.

To attempt to select representative instances from each

[1] Stokes, *Mathematical Papers*, p. 112.
[2] See p. 18.

recognised branch of mathematical inquiry would exceed the limits of this paper and the requirements of the argument. It must suffice, in conclusion, to direct attention to one species of Mathematics which seems largely affected with the property under consideration, the Calculus of Maxima and Minima, or (in a wide sense) *of Variations*. The criterion of a *maximum*[1] turns, not upon the *amount*, but upon the *sign* of a certain quantity.[2] We are continually concerned[3] with the ascertainment of a certain *loose quantitative relation*, the *decrease-of-rate-of-increase* of a quantity. Now, this is the very quantitative relation which it is proposed to employ in mathematical sociology; given in such data as the *law of diminishing returns to capital and labour*, the *law of diminishing utility*, the *law of increasing fatigue*; the very same irregular, unsquared material which constitutes the basis of the Economical and the Utilitarian Calculus.

Now, it is remarkable that the principal inquiries in Social Science may be viewed as *maximum-problems*. For Economics investigates the arrangements between agents each tending to his own *maximum* utility; and Politics and (Utilitarian) Ethics investigate the arrangements which conduce to the *maximum* sum total of

[1] *Maximum* in this paper is employed according to the context for (1) *Maximum* in the proper mathematical sense; (2) *Greatest possible*; (3) *stationary*; (4) where *minimum* (or *least possible*) might have been expected; upon the principle that every minimum is the correlative of a maximum. Thus Thomson's Minimum theorem is correlated with Bertrand's Maximum theorem. (Watson and Burbury.) This liberty is taken, not only for brevity, but also for the sake of a certain suggestiveness. '*Stationary*,' for instance, fails to suggest the *superlativeness* which it connotes.

[2] The second term of Variation. It may be objected that the *other* condition of a maximum equation of the first term to zero is of a more *precise* character. See, however, Appendix I., p. 92.

[3] E.g., Todhunter's *Researches on Calculus of Variations*, pp. 21–30, 80, 117, 286, &c.

utility. Since, then, Social Science, as compared with the Calculus of Variations, starts from similar data—*loose quantitative relations*—and travels to a similar conclusion—determination of *maximum*—why should it not pursue the same method, Mathematics?

There remains the objection that in Physical Calculus there is always (as in the example quoted above from Thomson and Tait) a potentiality, an expectation, of measurement; while Psychics want the first condition of calculation, *a unit*. The following[1] brief answer is diffidently offered.

Utility, as Professor Jevons[2] says, has two dimensions, *intensity* and *time*. The unit in each dimension is the just perceivable[3] increment. The implied equation to each other of each *minimum sensibile* is a first principle incapable of proof. It resembles the equation to each other of undistinguishable events or cases,[4] which constitutes the first principle of the mathematical calculus of *belief*. It is doubtless a principle acquired in the course of evolution. The implied equatability of time-intensity units, irrespective of distance in time and kind of pleasure, is still imperfectly evolved. Such is the unit of *economical* calculus.

For moral calculus a further dimension is required; to compare the happiness of one person with the happiness of another, and generally the happiness of groups of different members and different average happiness.

Such comparison can no longer be shirked, if there

[1] For a fuller discussion, see Appendix III.

[2] In reference to Economics, *Theory*, p. 51.

[3] Cf. Wundt, *Physiological Psychology*; below, p. 60. Our 'ebenmerklich' minim is to be regarded not as an infinitesimal differential, but as a finite small difference; a conception which is consistent with a (duly cautious) employment of infinitesimal notation.

[4] Laplace, *Essai—Probabilities*, p. 7.

is to be any systematic morality at all. It is postulated by distributive justice. It is postulated by the population question; that horizon in which every moral prospect terminates; which is presented to the far-seeing at every turn, on the most sacred and the most trivial occasions. You cannot spend sixpence utilitarianly, without having considered whether your action tends to increase the comfort of a limited number, or numbers with limited comfort; without having compared such alternative utilities.

In virtue of what *unit* is such comparison possible? It is here submitted: Any individual experiencing a unit of pleasure-intensity during a unit of time is to 'count for one.'[1] Utility, then, has *three* dimensions; a mass of utility, 'lot of pleasure,' is greater than another when it has more *intensity-time-number* units. The third dimension is doubtless an evolutional acquisition; and is still far from perfectly evolved.

Looking back at our triple scale, we find no peculiar difficulty about the third dimension. It is an affair of census. The second dimension is an affair of clock-work; assuming that the distinction here touched, between subjective and objective measure of time, is of minor importance. But the first dimension, where we leave the safe ground of the objective, equating to unity each *minimum sensibile*, presents indeed peculiar difficulties. *Atoms of pleasure* are not easy to distinguish and discern; more continuous than sand, more discrete than liquid; as it were nuclei of the just-perceivable, embedded in circumambient semi-consciousness.

We cannot *count* the golden sands of life; we cannot *number* the 'innumerable smile' of seas of love; but we

[1] In the Pure, for a *fraction*, in the Impure, imperfectly evolved, Utilitarianism. See p. 16.

seem to be capable of observing that there is here a *greater*, there a *less*, multitude of pleasure-units, mass of happiness; and that is enough.

(2) The application of mathematics to the world of soul is countenanced by the hypothesis (agreeable to the general hypothesis that every psychical phenomenon is the concomitant, and in some sense the other side of a physical phenomenon), the particular hypothesis adopted in these pages, that Pleasure is the concomitant of Energy. *Energy* may be regarded as the central idea of Mathematical Physics; *maximum energy* the object of the principal investigations in that science. By aid of this conception we reduce into scientific order physical phenomena, the complexity of which may be compared with the complexity which appears so formidable in Social Science.

Imagine a material Cosmos, a mechanism as composite as possible, and perplexed with all manner of wheels, pistons, parts, connections, and whose mazy complexity might far transcend in its entanglement the webs of thought and wiles of passion; nevertheless, if any given impulses be imparted to any definite points in the mechanism at rest, it is mathematically deducible that each part of the great whole will move off with a velocity such that the energy of the whole may be the greatest possible[1]—the greatest possible consistent with the given impulses and existing construction. If we know *something* about the construction of the mechanism, if it is 'a mighty maze, but not without a plan;' if we have some quantitative though not numerical datum about the construction, we may be able to deduce a similarly indefinite conclusion about the motion. For instance, any number of cases may be imagined in

[1] Bertrand's *Theorem*.

which, if a datum about the construction is that certain parts are *less stiff* than others, a conclusion about the motion would be that those parts [1] take on more energy than their stiffer fellows. This rough, indefinite, yet mathematical reasoning is analogous to the reasoning on a subsequent page,[2] that in order to the greatest possible sum total of happiness, the more capable of pleasure shall take more means, more happiness.

In the preceding illustration the motion of a mechanism was supposed instantaneously generated by the application of given impulses at definite points (or over definite surfaces); but similar general views are attainable in the not so dissimilar case in which we suppose motion generated in time by finite forces acting upon, and interacting between, the particles of which the mechanism is composed. This supposition includes the celebrated problem of Many Bodies (attracting each other according to any function of the distance); in reference to which one often hears it asked what can be expected from Mathematics in social science, when she is unable to solve the problem of Three Bodies in her own department. But Mathematics *can* solve the problem of many bodies—not indeed numerically and explicitly, but practically and philosophically, affording approximate measurements, and satisfying the soul of the philosopher with the grandest of generalisations. By a principle discovered or improved by Lagrange, each particle of the however complex whole is continually so moving that the accumulation of energy, which is constituted by adding to each other the energies of the mechanism existing at each instant of time (technically termed *Action*—the time-integral of Energy) should be a [3] maximum.

[1] Cf. Watson and Burbury, *Generalised Co-ordinates*, Art. 30, and preceding. [2] P. 64. [3] See note, p. 6.

By the discovery of Sir William Rowan Hamilton [1] the subordination of the parts to the whole is more usefully expressed, the velocity of each part is regarded as derivable from the *action* of the whole; the action is connected by a *single*, although not an explicit or in general easily interpretable, relation with the given law of force. The many unknown are reduced to one unknown, the one unknown is connected with the known.

Now this accumulation (or time-integral) of energy which thus becomes the principal object of the physical investigation is analogous to that accumulation of pleasure which is constituted by bringing together in prospect the pleasure existing at each instant of time, the end of rational action, whether self-interested or benevolent. The central conception of Dynamics and (in virtue of pervading analogies it may be said) in general of Mathematical Physics is *other-sidedly identical* with the central conception of Ethics; and a solution practical and philosophical, although not numerical and precise, as it exists for the problem of the interaction of bodies, so is possible for the problem of the interaction of souls.

This general solution, it may be thought, at most is applicable to the utilitarian problem of which the object is the greatest possible sum total of universal happiness. But it deserves consideration that an object of Economics also, the arrangement to which contracting agents actuated only by self-interest tend is capable of being regarded upon the psychophysical hypothesis here entertained as the realisation of the maximum sum-total of happiness, the *relative maximum*,[2] or that which is *consistent with certain conditions*. There is dimly discerned the Divine idea of a power tending to

[1] *Philosophical Transactions*, 1834-5. [2] See pp. 24, 142.

the greatest possible quantity of happiness [1] *under conditions*; whether the condition of that perfect disintegration and unsympathetic isolation abstractedly assumed in Economics, or those intermediate [2] conditions of what Herbert Spencer might term integration on to that perfected utilitarian sympathy in which the pleasures of another are accounted equal with one's own. There are diversities of conditions, but one maximum-principle; many stages of evolution, but 'one increasing purpose.'

'Mécanique Sociale' may one day take her place along with 'Mécanique Celeste,' throned each upon the double-sided height of one maximum principle,[3] the supreme pinnacle of moral as of physical science. As the movements of each particle, constrained or loose, in a material cosmos are continually subordinated to one maximum sum-total of accumulated energy, so the movements of each soul, whether selfishly isolated or linked sympathetically, may continually be realising the maximum energy of pleasure, the Divine love of the universe.

'Mécanique Sociale,' in comparison with her elder sister, is less attractive to the vulgar worshipper in that she is discernible by the eye of faith alone. The statuesque beauty of the one is manifest; but the fairy-like features of the other and her fluent form are

[1] Cf. Mill, *Essays on Nature and Religion*.

[2] See p. 16.

[3] The mathematical reader does not require to be reminded that upon the principles of Lagrange the whole of (conservative) Dynamics may be presented as a Maximum-Problem; if without gain, at any rate without loss. And the great principle of Thomson (Thomson & Tait, arts. Cf. *Theory of Vortices*, by Thomson, Royal Society, Edinburgh, 1865), with allied *maximum-principles*, dominating the theory of fluid motion, dominates Mathematical Physics with a more than nominal supremacy, and most indispensably efficacious power. Similarly, it may be conjectured, the ordinary moral rules are *equivalently* expressed by the Intuitivist in the (grammatically-speaking), *positive* degree, by the Utilitarian in the *superlative*. But for the higher moral problems the conception of *maximum* is indispensable.

veiled. But Mathematics has long walked by the evidence of things not seen in the world of atoms (the methods whereof, it may incidentally be remarked, statistical and rough, may illustrate the possibility of social mathematics). The invisible energy of electricity is grasped by the marvellous methods of Lagrange;[1] the invisible energy of pleasure may admit of a similar handling.

As in a system of conductors carrying electrical currents the energy due to electro-magnetic force is to be distinguished from the energy due to ordinary dynamical forces, *e.g.*, gravitation acting upon the conductors, so the energy of pleasure is to be distinguished not only from the gross energy of the limbs, but also from such nervous energy as either is not all represented in consciousness (*pace* G. H. Lewes), or is represented by *intensity of consciousness* not *intensity of pleasure*. As electro-magnetic force tends to a maximum energy, so also pleasure force tends to a maximum energy. The energy generated by pleasure force is the physical concomitant and measure of the conscious feeling of delight.

Imagine an electrical circuit consisting of two rails isolated from the earth connected at one extremity by a galvanic battery and bridged over at the other extremity by a steam-locomotive.[2] When a current of electricity is sent through the circuit, there is an electro-magnetic force tending to move the circuit or any moveable part of it in such a direction that the number of lines of force (due to the magnetism of the earth) passing through the circuit in a positive direction may be *a maximum*. The electro-magnetic force therefore tends to move the

[1] See Clerk Maxwell, *Electricity and Magnetism*, on the use of Lagrange's *Generalised Co-ordinates*, Part iv., chaps. 5 and 6.

[2] Clerk Maxwell has a similar construction.

locomotive along the rails in that direction. Now this delicate force may well be unable to move the ponderous locomotive, but it may be adequate to press a spring and turn a handle and let on steam and cause the locomotive to be moved by the steam-engine *in the direction of the electro-magnetic force*, either backwards or forwards according to the direction in which the electrical current flows. The delicate electro-magnetic force is placed in such a commanding position that she sways the movements of the steam-engine so as to satisfy her own yearning towards *maximum*.

Add now another degree of freedom; and let the steam-car governed move upon a *plane*[1] in a direction tending towards the position of Minimum Potential Electro-Magnetic Energy. Complicate this conception; modify it by substituting for the principle of Minimum Force-Potential the principle of *Minimum*[2] *Momentum-Potential*; imagine a comparatively gross mechanism of innumerable degrees of freedom *governed*, in the sense adumbrated, by a more delicate system—itself, however inconceivably diversified its degrees of freedom, obedient still to the great *Maximum Principles* of Physics, and amenable to mathematical demonstration, though at first sight as hopelessly incalculable as whatever is in life capricious and irregular—as the smiles of beauty and the waves of passion.

Similarly pleasure in the course of evolution has become throned among grosser subject energies—as it were explosive engines, ready[3] to go off at the pressure of a hair-spring. Swayed by the first principle, she sways the subject energies so as to satisfy her own yearning towards *maximum*; 'her every air Of gesture and least motion' a law of Force to governed systems—a fluent form, a Fairy Queen guiding a most complicated chariot, wheel within wheel, the 'speculative and active instruments,' the motor nerves, the limbs and the environment on which they act.

A system of such charioteers and chariots is what constitutes the object of Social Science. The attractions between the charioteer forces, the collisions and compacts between the chariots, present an appearance of quantitative regularity in the midst of bewildering complexity resembling in its general characters the field of electricity and magnetism. To construct a scientific hypothesis seems rather to surpass the powers of the writer than of Mathematics. 'Sin has ne possim naturæ accedere partes Frigidus obstiterit circum præcordia sanguis;' at least *the conception of Man as a pleasure machine* may justify and facilitate the employment of mechanical terms and Mathematical reasoning in social science.

[1] See p. 24.

[2] *Momentum-Potential* upon the analogy of *Velocity-Potential* (Thomson on Vortex Motion, § 31); and *Minimum*, as I venture to think, in virtue of certain analogies between theories about *Energy* and about *Action*.

[3] See the account of the *Mechanism of Life*, in Balfour Stewart's *Conservation of Energy*.

PART II.

SUCH are some of the preliminary considerations by which emboldened we approach the two fields into which the Calculus of Pleasure may be subdivided, namely Economics and Utilitarian Ethics. The Economical Calculus investigates the equilibrium of a system of hedonic forces each tending to maximum individual utility; the Utilitarian Calculus, the equilibrium of a system in which each and all tend to maximum universal utility. The motives of the two species of agents correspond with Mr. Sidgwick's Egoistic and Universalistic Hedonism. But the correspondence is not perfect. For, firstly, upon the principle of 'self limitation' of a method, so clearly stated by Mr. Sidgwick, so persistently misunderstood by critics, the Pure Utilitarian might think it most beneficent to sink his benevolence towards competitors; and the *Deductive Egoist* might have need of a Utilitarian Calculus. But further, it is possible that the moral constitution of the concrete agent would be neither Pure Utilitarian nor Pure Egoistic, but *μικτή τις*. For it is submitted that Mr. Sidgwick's division of Hedonism—the class of 'Method' whose principle of action may be generically defined *maximising happiness* —is not exhaustive. For between the two extremes Pure Egoistic and Pure Universalistic there may be an indefinite number of impure methods; wherein the happiness of others as compared by the agent (in a calm moment) with his own, neither counts for nothing, not yet 'counts for one,' but *counts for a fraction.*

Deferring controversy,[1] let us glance at the elements of the *Economic Calculus*; observing that the *connotation* (and some of the reasoning) extends beyond the usual denotation; to the political struggle for power, as well as to the commercial struggle for wealth.

ECONOMICAL CALCULUS.

DEFINITIONS.—The first principle of Economics[2] is that every agent is actuated only by self-interest. The workings of this principle may be viewed under two aspects, according as the agent acts *without*, or *with*, the

[1] See Appendix IV.

[2] *Descriptions* rather, but sufficient for the purpose of these tentative studies.

consent of others affected by his actions. In wide senses, the first species of action may be called *war*; the second, *contract*. Examples: (1) A general, or fencer, making moves, a dealer lowering price, *without consent of rival.* (2) A set of co-operatives (labourers, capitalists, manager) agreed *nem. con.* to distribute the joint-produce by assigning to each *a certain function* of his sacrifice. The *articles* of contract are in this case the *amount* of sacrifice to be made by each, *and the principle of distribution.*

'Is it peace or war?' asks the lover of 'Maud,' of economic *competition*, and answers hastily: It is both, *pax* or *pact* between contractors during contract, *war*, when some of the contractors *without the consent of others recontract.* Thus an auctioneer having been in contact with the last bidder (to sell at such a price *if* no higher bid) *recontracts* with a higher bidder. So a landlord on expiry of lease recontracts, it may be, with a new tenant.

The *field of competition* with reference to a contract, or contracts, under consideration consists of all the individuals who are willing and able to recontract about the articles under consideration. Thus in an auction the field consists of the auctioneer and all who are effectively willing to give a higher price than the last bid. In this case, as the transaction reaches determination, the field continually diminishes and ultimately vanishes. But this is not the case in general. Suppose a great number of auctions going on at the same point; or, what comes to the same thing, a market consisting of an indefinite number of dealers, say Xs, in commodity x, and an indefinite number of dealers, say Ys, in commodity y. In this case, up to the determination of equilibrium, the field continues indefinitely large. To be sure it may be said to vanish at the position of equilibrium. But that circumstance does not stultify the definition. Thus, if one chose to define the *field of force* as the centres of force sensibly acting on a certain system of bodies, then in a continuous medium of attracting matter, the field might be continually of indefinite extent, might change as the system moved, might be said to vanish when the system reached equilibrium.

There is free communication throughout a *normal* competitive field. You might suppose the constituent individuals collected at a point, or connected by telephones—an ideal supposition, but sufficiently approximate to existence or tendency for the purposes of abstract science.

A *perfect* field of competition professes in addition certain properties peculiarly favourable to mathematical calculation; namely, a certain indefinite *multiplicity* and *dividedness*, analogous to that *infinity* and *infinitesimality* which facilitate so large a portion of Mathematical Physics (consider the theory of Atoms, and all applications of the Differential Calculus). The conditions of a *perfect* field are four; the first pair referrible[1] to the heading *multiplicity* or continuity, the second to *dividedness* or fluidity.

I. Any individual is free to *recontract* with any out of an indefinite number, *e.g.*, in the last example there are an indefinite number of Xs and similarly of Ys.

II. Any individual is free to *contract* (at the same time) with an indefinite number; *e.g.*, any X (and similarly Y) may deal with any number of Ys. This condition combined with the first appears to involve

[1] See p. 5.

the indefinite divisibility of[1] each *article* of contract (if any X deal with an indefinite number of Ys he must give each an indefinitely small portion of x); which might be erected into a separate condition.

III. Any individual is free to *recontract* with another independently of, *without the consent* being required of, any third party, *e.g.*, there is among the Ys (and similarly among the Xs) no *combination* or precontract between two or more contractors that none of them will recontract without the consent of all. Any Y then may accept the offer of any X irrespectively of other Ys.

IV. Any individual is free to *contract* with another independently of a third party; *e.g.*, in simple exchange each contract is between two only, but *secus* in the entangled contract described in the example (p. 17), where it may be a condition of production that there should be three at least to each bargain.

There will be observed a certain similarity between the relation of the first to the second condition, and that of the third to the fourth. The failure of the first involves the failure of the second, but not *vice versâ*; and the third and fourth are similarly related.

A *settlement* is a contract which cannot be varied with the consent of all the parties to it.

A *final settlement* is a settlement which cannot be varied by recontract within the field of competition.

Contract is *indeterminate* when there are an indefinite number of *final settlements*.

[1] This species of imperfection will not be explicitly treated here; partly because it is perhaps of secondary practical importance; and partly because it has been sufficiently treated by Prof. Jevons (*Theory*, pp. 135-137). It is important, as suggested in Appendix V., to distinguish the effects of this imperfection according as the competition is, or is not, supposed perfect in *other* respects.

The PROBLEM to which attention is specially directed in this introductory summary is: *How far contract is indeterminate*—an inquiry of more than theoretical importance, if it show not only that indeterminateness tends to prevent widely, but also in what direction an escape from its evils is to be sought.

DEMONSTRATIONS.[1]—The general answer is—(α) Contract without competition is indeterminate, (β) Contract with *perfect* competition is perfectly determinate, (γ) Contract with more or less perfect competition is less or more indeterminate.

(α) Let us commence with almost the simplest case of contract,—two individuals, X and Y, whose interest depends on two variable quantities, which they are agreed not to vary without mutual consent. Exchange of two commodities is a particular case of this kind of contract. Let x and y be the portions interchanged, as in Professor Jevons's example.[2] Then the utility of one party, say X, may be written $\Phi_1(a-x) + \Psi_1(y)$; and the utility of the other party, say Y, $\Phi_2(x) + \Psi_2(b-y)$; where Φ and Ψ are the integrals of Professor Jevons's symbols ϕ and ψ. It is agreed that x and y shall be varied only by consent (not *e.g.* by violence).

More generally. Let P, the utility of X, one party, $= \mathrm{F}(x\,y)$, and Π, the utility of Y, the other party, $= \Phi(x\,y)$. If now it is inquired at what point they will reach equilibrium, one or both refusing to move further, to what *settlement* they will consent; the answer is in general that contract by itself does not supply sufficient conditions to determinate the solution; supplementary conditions as will appear being supplied by

[1] *Conclusions* rather, the mathematical demonstration of which is not fully exhibited.

[2] *Theory of Political Economy*, 2nd ed., p. 107.

competition or ethical motives, Contract will supply only *one* condition (for the two variables), namely

$$\frac{d\mathrm{P}}{d\,x}\frac{d\,\Pi}{d\,y} = \frac{d\mathrm{P}}{d\,y}\frac{d\,\Pi}{d\,x}$$

(corresponding to Professor Jevons's equation

$$\frac{\phi_1(a-x)}{\psi_1(y)} = \frac{\phi_2(x)}{\psi_2(b-y)}$$

Theory p. 108), which it is proposed here to investigate.

Consider $\mathrm{P} - \mathrm{F}(x\,y) = 0$ as a surface, P denoting the length of the ordinate drawn from any point on the plane of $x\,y$ (say the plane of the paper) to the surface. Consider $\Pi - \Phi(x\,y)$ similarly. It is required to find a point $(x\,y)$ such that, *in whatever direction* we take an infinitely small step, P and Π do not increase together, but that, while one increases, the other decreases. It may be shown from a variety of points of view that the locus of the required point is

$$\frac{d\mathrm{P}}{d\,x}\,\frac{d\,\Pi}{d\,y} - \frac{d\mathrm{P}}{d\,y}\,\frac{d\,\Pi}{d\,x} = 0\,;$$

which locus it is here proposed to call the *contract-curve*.

(1) Consider first in what directions X can take an indefinitely small step, say of length ρ, from any point $(x\,y)$. Since the addition to P is

$$\rho\left[\left(\frac{d\mathrm{P}}{d\,x}\right)\cos\theta + \left(\frac{d\mathrm{P}}{dy}\right)\sin\,\theta\right],$$

$\rho\cos\theta$ being $= d\,x$, and $\rho\sin\theta = d\,y$, it is evident that X will step only on one side of a certain line, the *line of indifference*, as it might be called; its equation being

$$(\zeta - x)\left(\frac{d\mathrm{P}}{d\,x}\right) + (\eta - y)\left(\frac{d\,\mathrm{P}}{d\,y}\right) = 0.$$

And it is to be observed, in passing, that the direction in which X will *prefer* to move, the line of force or *line of preference*, as it may be termed, is perpendicular to the line of indifference. Similar remarks apply to Π. If then we enquire in what directions X and Y will consent to move *together*, the answer is, in any direction between their respective lines of indifference, in a direction *positive* as it may be called *for both*. At what point then will they refuse to move at all? When their *lines of indifference* are coincident (and *lines of preference* not only coincident, but in opposite directions); whereof the *necessary* (but *not sufficient*) condition is

$$\left(\frac{dP}{dx}\right)\left(\frac{d\Pi}{dy}\right)-\left(\frac{dP}{dy}\right)\left(\frac{d\Pi}{dx}\right) = 0.$$

(2) The same consideration might be thus put. Let the complete variation of P be $DP=\rho\left[\left(\frac{dP}{dx}\right)\cos\theta + \left(\frac{dP}{dy}\right)\sin\theta\right]$ and similarly for Π. Then in general θ can be taken, so that $\frac{DP}{D\Pi}$ should be positive, say $= g^2$, and so P and Π both increase together.

$$\tan.\ \theta = -\frac{\frac{dP}{dx} - g^2\frac{d\Pi}{dx}}{\frac{dP}{dy} - g^2\frac{d\Pi}{dy}}$$

But this solution fails when $\frac{\left(\frac{dP}{dx}\right)}{\left(\frac{dP}{dy}\right)} = \frac{\left(\frac{d\Pi}{dx}\right)}{\left(\frac{d\Pi}{dy}\right)}$

In fact, in this case $\frac{DP}{D\Pi}$ *is the same for all directions.*

If, then, that common value of $\frac{DP}{D\Pi}$ is *negative*, motion is impossible in any direction.

(3) Or, again, we may consider that motion is possible so long as, one party not losing, the other gains. The point of equilibrium, therefore, may be described as a *relative maximum*, the point at which *e.g.* Π being constant, P is a maximum. Put $P = P - c(\Pi - \Pi')$, where c is a constant and Π' is the supposed given value of Π. Then P is a maximum only when

$$dx\left(\frac{dP}{dx} - c\frac{d\Pi}{dx}\right) + dy\left(\frac{dP}{dy} - c\frac{d\Pi}{dy}\right) = 0;$$

whence we have as before the *contract-curve.*

The same result would follow if we supposed Y induced to consent to the variation, not merely by the guarantee that he should not lose, or gain infinitesimally, but by the understanding that he should gain sensibly with the gains of P. For instance, let $\Pi = k^2P$ where k is a constant, certainly not a very practicable condition. Or, more generally, let P move subject to the condition that $DP = \theta^2 \times D\Pi$, where θ is a function of the coordinates. Then DP, *subject to this condition*, vanishes only when

$$0 = \left(\frac{dP}{dx}\right)dx + \left(\frac{dP}{dy}\right)dy + c\left\{\left(\frac{dP}{dx}\right)dx + \left(\frac{dP}{dy}\right)dy - \theta^2\left[\left(\frac{d\Pi}{dx}\right)dx + \left(\frac{d\Pi}{dy}\right)dy\right]\right\}$$

where c is a constant;

$$\text{whence}\left(\frac{dP}{dx}\right)(1+c) - c\theta^2\left(\frac{d\Pi}{dx}\right) = 0$$

and $$\left(\frac{dP}{dx}\right)(1+c) - c\theta^2\left(\frac{d\Pi}{dy}\right) = 0;$$