Common Agency without Delegation

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Abstract

In order to address Szentes' critique (Szentes (2009)), we study a common-agency-without-delegation model. We prove that the menu theorem in Peters (2001) holds only partially in our model under some particular information structure. We use examples to show that it fails generally. Finally, we prove a menu-of-menu-with-recommendation theorem in our model.

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1 Introduction

In a canonical principal-agent model, before the principal takes an action, she[1] can design a mechanism (or equivalently, contract) to elicit the payoff-relevant information which is held privately by the agent. Given a goal, the principal aims to solve a mechanism design problem, i.e., she aims to find an optimal mechanism among the set of all possible mechanisms, in order to achieve the goal. This is a very difficult problem, because the set of all possible mechanisms is complicated.

The Revelation Principle is a milestone in mechanism design, which says that it suffers no loss of generality for the principal to focus only on (1) the set of direct mechanisms[2] and on (2) the truth-revealing equilibrium. As a result, the Revelation Principle transforms a complicated problem into a much more tractable problem. Clearly, the revelation principle has two indispensable parts (i.e., (1) and (2) above), even though it is named after (2).

However, the Revelation Principle fails, when multiple principals exist.[3] Thus, it has been a long-time open question regarding equilibrium characterization in competing mechanism games.

In a common-agency model, we have multiple principals and one agent. Though the Revelation Principle fails in a common-agency model, Peters (2001) (and Martimort and Stole (2002)) prove a menu theorem (or equivalently, a taxation principle). Specifically, they identify a much simpler contract space, called the set of menus of actions, and prove that in common-agency models, it suffers no loss of generality for each principal to offer a menu of her actions, letting the agent choose an action in the menu.

In a competing mechanism game with multiple principals and multiple agents, the menu theorem does not hold (see the discussion in Peters (2001)). Instead, Yamashita

[1]Throughout the paper, we use she to denote a principal, and he to denote an agent.

[2]In a direct mechanism, the principal invites agents to reveal their payoff types that characterize their preference relations. The reports on payoff types fully determine the action chosen by the principal.

[3]If there is only one principal, an agent's decision depends only on his payoff type. If there are multiple principals, an agent's decision depends not only on his payoff type but also the contracts offered by all the principals. Thus, it suffers loss of generality to invite an agent to reveal his payoff type only. In addition, the agent must report market information on the contracts offered by all the principals.

(2010) develops a folk theorem to characterize equilibrium allocations. Szentes (2009) raises a critique on Yamashita (2010), and shows that the folk theorem hinges critically on the assumption that principals delegate their decisions to agents (i.e., agents' messages fully determines a principal's action in a contract). Szentes (2009) argues that principals should offer non-delegated contracts, in which agents' messages determines only a subset of actions from which a principal subsequently chooses her action.

As in Yamashita (2010), Peters (2001) proves the menu theorem under the assumption that principals delegate their decisions to the agent. This immediately leads to the following question: if principals offer non-delegate contracts, does the menu theorem still hold? In this paper, we rigorously define four common-agency-without-delegation models, and we aim to answer this question for all of the four models.

A common-agency game without delegation consists of three stages.

$$\left(\begin{array}{c}\text{Stage 1: principals simultaneously announce their non-delegated contracts to the agent;}\\ \text{Stage 2: the agent simultaneously sends messages to principals,}\\ \text{which pin down a subset of actions for each principal;}\\ \text{Stage 3: each principal simultaneously chooses an action in the subset.}\end{array}\right)$$

Different from the delegated model in Peters (2001), the announcement and communication structures matter in our non-delegated model because a principal's action choice at Stage 3 depends on what she has observed at the time. Specifically, at Stage 1, each principal may announce her contract to the agent privately or publicly. With public announcement, principals observe all of the contracts announced by all principals, whereas with private announcement, each principal observes only her own contract. Furthermore, at Stage 2, the agent may send a message to each principal privately or publicly. With public communication, principals observe all of the agent's messages sent to all principals, whereas with private communication, each principal observes only the message she receives. That is, with different combinations of announcement and communication protocols, we can define four non-delegated common-agency models.[4]

$$\begin{pmatrix} \text{Model 1: private announcement and private communication;} \\ \text{Model 2: public announcement and private communication;} \\ \text{Model 3: private announcement and public communication;} \\ \text{Model 4: public announcement and public communication} \end{pmatrix}.$$

In Section 2, we show that Szentes' critique remains a valid critique in common agency although it is originally raised in the multiple-principal-multiple-agent model. In Section 4.2, we construct examples to show that if delegated contracts are allowed, it suffers loss of generality to focus on menu contracts because there are new equilibrium allocations that cannot be reproduced by equilibria in the menu game in Models 2, 3 and 4.[5]

In Section 6, Proposition 1 shows that the menu theorem holds only *partially* in Model 1 in the sense that principals can focus on menu contracts only on the equilibrium path, but this is not true on off-equilibrium paths.

Our main contribution is to identify two simple contract spaces principals can use on and off the paths respectively without loss of generality, which leads to the full characterization of equilibrium allocations in all four models. A natural guess for such simple contract spaces is the menu-of-menu contract, i.e., a principal offers a menu of subsets of actions, letting the agent choose one subset of actions in the menu. But the menu-of-menu contract does not work for our purposes. In an equilibrium involving complex non-delegated contracts c^*, the agent, at two different states θ and θ', may send two different messages, m_j and m'_j to principal j, which induce the same subset of actions (i.e., $c^*_j(m_j) = c^*_j\left(m'_j\right)$), but two different equilibrium actions, y_j and y'_j, are chosen by principal j at Stage 3 (in reponse to m_j and m'_j, repectively). However, if we use menu-of-menu contracts to replicate c^*, the agent, at two different states θ and θ', must send the same messages, $c^*_j(m_j) = c^*_j\left(m'_j\right)$, to principal j, which would necessarily induce the same action at Stage 3 (in reponse to the same message $c^*_j(m_j) = c^*_j\left(m'_j\right)$).

[4]For the delegated common-agency model in Peters (2001), the annoucement and communication protocols do not have impact on equilibria.

[5]In Model 1, any equilibrium allocation in the non-delegated contract game can be reproduced by the menu game (See Theorem 3 in Section 6), but not every equilibrium in the menu game survives if non-delegated contracts are allowed for principals' deviations.

Not only do menu-of-menu contracts not work, but also the (minimal) contract spaces that work on and off the equilibrium paths are different, which are described as follows.

$$\left(\begin{array}{c} \text{on-equilibrium paths: menu-of-menu-with-recommendation contracts (Definition 3);} \\ \text{off-equilibrium paths: menu-of-menu-with-full-recommendation contracts (Definition 4)} \end{array} \right).$$

A menu-of-menu-with-recommendation contract is a menu of combinations of a subset of actions and a recommeded action within the subset. In this contract, a principal can specify what actions in a subset of actions the agent can recommend for her. Nevertheless, the recommeded action is non-binding. For example, the following is a menu-of-menu-with-recommendation contract that principal j may offer.

$$\begin{gathered} c_j : K_j \to 2^{Y_j} \diagdown \{\varnothing\}, \\ K_j \equiv \{[\{a,b\},a], [\{c,d\},c], [\{c,d\},d], \ [\{e,f,g\},e], \ [\{e,f,g\},f]\}, \\ c_j([\{a,b\},a]) = \{a,b\} \text{ and } c_j([\{c,d\},c]) = c_j([\{c,d\},d]) = \{c,d\}. \\ c_j([\{e,f,g\},e]) = c_j([\{e,f,g\},f]) = \{e,f,g\} \end{gathered}$$

A menu-of-menu-with-full-recommendation contract is a special menu-of-menu-with-recommendation contract. Specifically, it offers a menu of subsets of actions, and there exists a unique subset such that every action in the subset must be recommended by the agent, and all the other subsets gets no recommendation.[6] For example, the following is a menu-of-menu-with-recommendation contract that principal j may offer.

$$\begin{gathered} \widehat{c}_j : \widehat{K}_j \to 2^{Y_j} \diagdown \{\varnothing\}, \\ \widehat{K}_j \equiv \{\{a,b\}, \ \{c,d,e\}, \ [\{f,g,h\},f], \ [\{f,g,h\},g], \ [\{f,g,h\},h]\}, \\ \widehat{c}_j(\{a,b\}) = \{a,b\}, \widehat{c}_j(\{c,d,e\}) = \{c,d,e\}, \\ \widehat{c}_j([\{f,g,h\},f]) = \widehat{c}_j([\{f,g,h\},g]) = \widehat{c}_j([\{f,g,h\},h]) = \{f,g,h\}. \end{gathered}$$

We prove that, in models 2, 3 and 4, it suffers no loss of generality to focus on menu-of-menu-with-recommendation contracts on equilibrium paths. Furthermore, in models 1, 2, 3 and 4, it suffers no loss of generality to focus on menu-of-menu-with-full-recommendation contracts on off-equilibrium paths.

[6] Or equivalently, since recommendation is non-binding, we can assign an arbitrary single recommendation for each of the other subsets.

The remainder of the paper proceeds as follows: we review Szentes' critique in Section 2; we describe the model in Section 3; we show failure of the menu theorem in Section 4; we define two simpler contract spaces in Section 5; we provide full equilibrium characterization in Sections 6 and 7. We conclude in Section 9.

2 Szentes' critique

Consider the following example in Szentes (2009). There are two principals and three agents. There is only one payoff-relevant state, i.e., we have complete information. The two principals must choose one of the two actions, H and T. The following table describes principals' utility for each action profile.

	H	T
H	$1,-1$	$-1,1$
T	$-1,1$	$1,-1$

All agents are indifferent among all action profiles.

In this game, each principal has a min max value of 1 and a max min value of -1. For instance, (T,H) induces the max min value for principal 1 (i.e., the row player). By the folk theorem in Yamashita (2010), (T,H) can be induced by an equilibrium. Specifically, on the equilibrium path, each principal j offers the contract $c_j : \{H,T\} \times \{H\} \times \{H\} \to \{H,T\}$ with

$$c_j(H,H,H) = H \text{ and } c_j(T,H,H) = T.$$

The interpretation is that principals invite agent 1 only to vote regarding "H vs T,"[7] and principals follows agent 1's recommendation. Note that c_j described above is a delegated contract because agents' messages fully determines principal j's action. Upon receiving such a contract profile on the equilibrium path, agent 1 votes T and H for principals 1 and 2, respectively, which induces (T,H).

We now show that the strategy profile described above is an equilibrium. First, agents are indifferent among all action profiles, and hence, their incentive compatibility always holds. Second, (T,H) induces the maximal utility for principal 2, and hence,

[7] The message sets for agents 2 and 3 are degenerate (i.e., $|\{H\}| = 1$), and their votes are not informative.

incentive compatibility of principal 2 holds. Third, if principal 1 deviates unilaterally to any contract, in which $y_1 \in \{H, T\}$ is induced by some message $\widehat{m}$. Given this unilateral deviation, a continuation equilibrium is that agents send $\widehat{m}$ to principal 1 and agent 1 recommends $y_2 \in \{H, T\} \diagdown \{y_1\}$ to principal 2. This would induce the max min value for principal 1, i.e., this deviation is not profitable for principal 1.

However, Szentes (2009) doubts the legitimacy of (T, H) being an equilibrium outcome. Szentes (2009) argues that, in any reasonable equilibrium, every principal must achieve at least her min max value, because she can always opt out of this contract game (and achieves her min max value in the ensuing equilibrium).[8] Furthermore, Szentes (2009) identifies the source of this problem: the delegated contract, i.e., agents' messages pinning down principals' actions. Rigorously, Szentes (2009) argues that principals should offer non-delegated contracts (i.e., agents' messages pinning down a subset of actions for each principal), and proves that each principal indeed achieves her min max value under non-delegated contracts.

Though there are multiple agents in the example above, the same logic still applies if we delete agents 2 and 3 from the model. Therefore, Szentes' critique also applies to the common-agency model.

The menu theorem in Peters (2001) is proved under the assumption that principals offer delegated contracts only. This immediately leads to the question: does the menu theorem still holds, if principals offer non-delegated contracts? In order to answer this question, we first rigorously define a common-agency-without-delegation model in the next section.

3 Preliminaries

A single agent is privately informed about her type $\theta \in \Theta$, which is drawn from a common prior $p \in \Delta(\Theta)$ with full support. Throughout the paper, we assume $|\Theta| < \infty$. Let $\mathcal{J} \equiv \{1, \dots, J\}$ be the set of principals, i.e., there are J principals. Each principal j takes her own action $y_j \in Y_j$. Let $Y \equiv \times_{j \in \mathcal{J}} Y_j$. Principal j's utility function is $v_j : Y \times \Theta \to \mathbb{R}$.

[8] See more discussions in Peters (2014).

The agent's utility function is $u : Y \times \Theta \to \mathbb{R}$.

For each $j \in \mathcal{J}$, principal j's contract is a function $c_j : M_j^{\mathcal{A}} \to \mathcal{A}_j$, where $M_j^{\mathcal{A}}$ is the set of messages that the agent can send to principal j and

$$\mathcal{A}_j^{delegated} \equiv \left\{ \{y_j\} : y_j \in Y_j \right\},$$
$$\mathcal{A}_j^{non-delegated} \equiv 2^{Y_j} \diagdown \{\varnothing\},$$
$$\mathcal{A}_j \in \left\{ \mathcal{A}_j^{delegated}, \mathcal{A}_j^{non-delegated} \right\}.$$

$M_j^{\mathcal{A}}$ can be very general (and complex) and we do not impose any restrictions on it.[9] Let $M^{\mathcal{A}} \equiv \times_{j \in \mathcal{J}} M_j^{\mathcal{A}}$ and $M_{-j}^{\mathcal{A}} \equiv \times_{k \neq j} M_k^{\mathcal{A}}$. If $\mathcal{A}_j = \mathcal{A}_j^{delegated}$, principal j delegates her action choice to the agent in that the agent's message fully determines her action. If $\mathcal{A}_j = \mathcal{A}_j^{non-delegated}$, principal j has to make a non-degenerate strategic action choice in $c_j(m_j) \in 2^{Y_j} \diagdown \{\varnothing\}$ after receiving a message m_j. Define $\mathcal{A} \equiv \times_{j \in \mathcal{J}} \mathcal{A}_j$, and $\mathcal{A}_{-j} \equiv \times_{k \neq j} \mathcal{A}_k$. Given $\mathcal{A}_j$, let $C_j^{\mathcal{A}}$ be the set of all possible contracts available to principal j, $C^{\mathcal{A}} \equiv \times_{j \in \mathcal{J}} C_j^{\mathcal{A}}$ and $C_{-j}^{\mathcal{A}} \equiv \times_{k \neq j} C_k^{\mathcal{A}}$.

3.1 Models and timeline

When principal j chooses her action in the set $c_j\left(m_j\right)$, her action choice depends not only on the message she receives but also on what she observes regarding the contracts offered by the other principals and the messages that the agent sends to principals.

We denote the announcement structure by $\Gamma \equiv \left[\Gamma_j : C^{\mathcal{A}} \to 2^{C^{\mathcal{A}}}\right]_{j \in \mathcal{J}}$. In particular, we focus on two announcement structures. First, public announcement, denoted by $\Gamma^{public} = \times_{j \in \mathcal{J}} \Gamma_j^{public}$, is defined as follows.

$$\Gamma_j^{public}\left(c_j, c_{-j}\right) = \left\{ \left(c_j, c_{-j}\right) \right\}, \forall j \in \mathcal{J}, \forall \left(c_j, c_{-j}\right) \in C^{\mathcal{A}}. \tag{1}$$

Second, private announcement, denoted by $\Gamma^{private} = \times_{j \in \in \mathcal{J}} \Gamma_j^{private}$, is defined as follows.

$$\Gamma_j^{private}\left(c_j, c_{-j}\right) = \left\{c_j\right\} \times C_{-j}^{\mathcal{A}}, \forall j \in \mathcal{J}, \forall \left(c_j, c_{-j}\right) \in C^{\mathcal{A}}. \tag{2}$$

[9] In particular, the superscript $\mathcal{A}$ does not restrict the generality of communication in any way. It is purely for notational ease when we distinguish a general class of contracts from other classes of contracts that we will define later.

The interpretation is that, principals observe all of the contracts offered under public announcement, whereas, under private announcement, each principal knows only her own contract.

Similarly, we denote the communication structure by $\Psi \equiv \left[\Psi_j : M^{\mathcal{A}} \to 2^{M^{\mathcal{A}}}\right]_{j\in\mathcal{J}}$. In particular, we focus on two communication structures. First, public communication, denoted by $\Psi^{public} = \times_{j\in\mathcal{J}} \Psi_j^{public}$, is defined as follows.

$$\Psi_j^{public}\left(m_j, m_{-j}\right) = \left\{\left(m_j, m_{-j}\right)\right\}, \forall j \in \mathcal{J}, \forall \left(m_j, m_{-j}\right) \in M^{\mathcal{A}}. \tag{3}$$

Second, private communication, denoted by $\Psi^{private} = \times_{j\in\in\mathcal{J}} \Psi_j^{private}$, is defined as follows.

$$\Psi_j^{private}\left(m_j, m_{-j}\right) = \left\{m_j\right\} \times M_{-j}^{\mathcal{A}}, \forall j \in \mathcal{J}, \forall \left(c_j, c_{-j}\right) \in M^{\mathcal{A}}. \tag{4}$$

Thus, a model is characterized by a tuple $\langle \mathcal{A}, \Gamma, \Psi \rangle$, where

$$\mathcal{A} \in \left\{\mathcal{A}^{\text{delegated}}, \mathcal{A}^{\text{non-delegated}}\right\}, \Gamma \in \left\{\Gamma^{private}, \Gamma^{public}\right\}, \Psi \in \left\{\Psi^{private}, \Psi^{public}\right\}$$

Given a model $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle$, the common-agency game proceeds according to the following timeline.

1. Before the game starts, Nature draws the agent's type according to the prior $p \in \Delta(\Theta)$ and it is the agent's private information;
2. At Stage 1, each principal $j \in \mathcal{J}$ simultaneously offers a contract $c_j \in C_j^{\mathcal{A}}$ to the agent. The agent observes $c \equiv (c_1, \ldots, c_J)$, whereas each principal $j \in \mathcal{J}$ knows that only a contract profile in $\Gamma_j(c)$ is possibly chosen by principals.
3. At Stage 2, the agent sends messages $m \equiv (m_1, \ldots, m_J) \in M^{\mathcal{A}}$, one for each principal. Each principal $j \in \mathcal{J}$ knows that only a message profile in $\Psi_j(m)$ is possibly sent by the agent to principals;
4. At Stage 3, each principal $j \in \mathcal{J}$ simultaneously chooses an action in $c_j(m_j)$;
5. Finally, payoffs are realized.

For simplicity, we do not consider the agent's effort choice to provide a clear insight into our results, but we can add the agent's effort choice after he observes actions chosen by principals as the last stage of the game[10] (see discussion in Section 8.1).

3.2 Strategies

In a model $\langle \mathcal{A}, \Gamma, \Psi \rangle$, we define strategies for players to formulate the notion of pure-strategy *perfect Bayesian equilibrium*. Principal j's contract strategy at Stage 1 is $c_j \in C_j^{\mathcal{A}}$. Let $c \equiv (c_1, \ldots, c_J) \in C^{\mathcal{A}}$ and $c_{-j} \in C_{-j}^{\mathcal{A}}$. The agent's communication strategy at Stage 2 is a function $s \equiv [s_j : C^{\mathcal{A}} \times \Theta \to M_j^{\mathcal{A}}]_{j \in \mathcal{J}}$. Let S_j be the set of all possible $s_j : C^{\mathcal{A}} \times \Theta \to M_j^{\mathcal{A}}$. Let $s \equiv (s_1, \ldots, s_J) \in S \equiv \times_{j \in \mathcal{J}} S_j$ and $s_{-j} \in S_{-j} \equiv \times_{k \neq j} S_k$. For all $(c, \theta) \in C^{\mathcal{A}} \times \Theta$, $s(c, \theta) = (s_1(c, \theta), \ldots, s_J(c, \theta))$. Each principal j's action choice strategy is a function $t_j : \Gamma_j\left(C^{\mathcal{A}}\right) \times \Psi_j\left(M^{\mathcal{A}}\right) \to Y_j$ such that

$$t_j\left[\Gamma_j\left(c_j, c_{-j}\right), \Psi_j\left(m_j, m_{-j}\right)\right] \in c_j(m_j), \forall \left[\left(c_j, c_{-j}\right), \left(m_j, m_{-j}\right)\right] \in C^{\mathcal{A}} \times M^{\mathcal{A}}.$$

Let T_j be the set of all possible action choice strategies for principal j. Let $t \equiv (t_1, \ldots, t_J) \in T \equiv \times_{j \in \mathcal{J}} T_j$ and $t_{-j} \in T_{-j} \equiv \times_{k \neq j} T_k$

Fix a profile of strategies $(c, s, t) \in C^{\mathcal{A}} \times S \times T$. Given (c, s, t), the utility for the agent of type θ is

$$U(c, s, t, \theta) \equiv u\left(t_1\left(\Gamma_1(c), \Psi_1\left(s_1(c, \theta)\right)\right), \ldots, t_J\left(\Gamma_J(c), \Psi_J\left(s_1(c, \theta)\right)\right), \theta\right)$$

Given (c, s, t), principal j's expected utility is

$$V_j(c, s, t) \equiv \sum_{\theta \in \Theta} p(\theta)\, v_j\left(t_1\left(\Gamma_1(c), \Psi_1\left(s_1(c, \theta)\right)\right), \ldots, t_J\left(\Gamma_J(c), \Psi_J\left(s_1(c, \theta)\right)\right), \theta\right)$$

At Stage 3, principal j must form a belief on $\left(C^{\mathcal{A}} \times M^{\mathcal{A}} \times \Theta\right)$ conditional on $\Gamma_j(c)$ she observes at Stage 1 and $\Psi_j(m)$ she observes at Stage 2, which is described by a function $b_j : \Gamma_j\left(C^{\mathcal{A}}\right) \times \Psi_j\left(M^{\mathcal{A}}\right) \to \Delta\left(C^{\mathcal{A}} \times M^{\mathcal{A}} \times \Theta\right)$. Given belief b_j, principal j's expected utility conditional on $\left(\alpha_j, \beta_j\right) \in \Gamma_j\left(C^{\mathcal{A}}\right) \times \Psi_j\left(M^{\mathcal{A}}\right)$ is

$$V_j\left(t_j, t_{-j} | \alpha_j, \beta_j, b_j\right) \equiv \int_{b_j\left(\alpha_j, \beta_j\right)} v_j\left(t_1\left(\Gamma_1(c), \Psi_1(m)\right), \ldots, t_J\left(\Gamma_J(c), \Psi_J(m)\right), \theta\right) d(c, m, \theta).$$

[10] In this case, $y_j \in Y_j$ is an incentive schedule that specifies principal j's action conditional on each possible effort chosen by the agent, rather than principal j's action per se. Given a profile of incentive schedules, the agent choose his effort, which then determines principals' actions.

3.3 Legitimate beliefs

For each principal j's belief, we apply Bayes' rule only whenever she cannot confirm that the other players have deviated from an equilibrium. Given (c, s) that is played in an equilibrium, define $\mathcal{B}_j^{(c,\,s)}$ the set of principal j' valid beliefs in an equilibrium as follows: For all $j \in \mathcal{J}$

$$\mathcal{B}_j^{(c,\,s)} \equiv \left\{ \begin{array}{c} b_j : \Gamma_j\left(C^{\mathcal{A}}\right) \times \Psi_j\left(M^{\mathcal{A}}\right) \to \Delta\left(C^{\mathcal{A}} \times M^{\mathcal{A}} \times \Theta\right) \text{ such that} \\ \left(\begin{array}{c} \forall c_j' \in C_j^{\mathcal{A}_j},\ \forall \theta \in \Theta, \\ \text{set } \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta\right)\right), \text{ and we have} \\ b_j\left[\Gamma_j\left(c_j', c_{-j}\right),\ \beta_j\right]\left(c_j', c_{-j},\ s\left(c_j', c_{-j}, \theta\right),\ \theta\right) = \frac{p(\theta)}{\sum\limits_{\theta'' \in \left\{\theta' \in \Theta:\ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\}} p(\theta'')} \end{array} \right) \end{array} \right\}. \tag{5}$$

We will adopt the solution concept of (weak) Perfect Bayesian equilibrium, and hence, players will use Bayes' rule to update their beliefs whenever possible. As usual, when one principal deviates unilaterally, she assumes that the other players follows the equilibrium strategy profile. This is rigorously described in the set $\mathcal{B}_j^{(c,\,s)}$ in (5). Specifically, $\mathcal{B}_j^{(c,\,s)}$ contains any belief function b_j which satisfies the following condition. Given (c, s) that is played in an equilibrium, suppose principal j unilateral deviates to $c_j' \in C_j^{\mathcal{A}_j}$. If j observes $\Gamma_j\left(c_j', c_{-j}\right)$ (i.e., j cannot confirm that principals $-j$ have deviated from (c, s)) and observes $\beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta\right)\right)$ for some $\theta \in \Theta$ (i.e., j cannot confirm that agents have deviated from (c, s)), principal j believes that principals $-j$ have offered c_{-j}, and the agent has followed s. As a result, the following set contains all possible states,

$$\left\{\theta' \in \Theta:\ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\},$$

and by Bayes' rule, j's updated belief is

$$b_j\left[\Gamma_j\left(c_j', c_{-j}\right),\ \beta_j\right]\left(c_j', c_{-j},\ s\left(c_j', c_{-j}, \theta\right),\ \theta\right) = \frac{p(\theta)}{\sum\limits_{\theta'' \in \left\{\theta' \in \Theta:\ \beta_j = \Psi_j\left(s\left(c_j', c_{-j}, \theta'\right)\right)\right\}} p(\theta'')}.$$

If principal j can confirm that either principals $-j$ or the agent have deviated from (c, s), we impose no requirement on $b_j \in \mathcal{B}_j^{(c,\,s)}$, because this happens with probability 0 in an equilibrium, and Bayes rule does not apply. For instance, suppose $\Gamma_j = \Gamma_j^{public}$, and principal j observes $\Gamma_j^{public}\left(c'_j, c'_{-j}\right)$ with $c'_{-j} \neq c_{-j}$. Then, we impose no requirement on $b_j\left[\Gamma_j^{public}\left(c'_j, c'_{-j}\right), \beta_j\right]$ for every $\beta_j \in \Psi_j\left(M^{\mathcal{A}}\right)$.

3.4 Perfect Bayesian equilibrium

We adopt the solution concept of pure-strategy perfect Bayesian equilibrium in the most parts of the paper. For simplicity, we just call it an equilibrium. Our results can be extended for mixed-strategy equilibria (See Section 8.2). By backward induction, we first suppose that $c \in C^{\mathcal{A}}$ is the common prior belief on principals' equilibrium contracts and define a continuation equilibrium.

Definition 1 *In a model $\langle \mathcal{A}, \Gamma, \Psi \rangle$, let $c \in C^{\mathcal{A}}$ be the common prior belief on principals' equilibrium contracts. $[s, t]$ is a continuation equilibrium if there exists*

$$b_j \in \mathcal{B}_j^{(c,s)}, \forall j \in \mathcal{J},$$

such that (i) for all $j \in \mathcal{J}$,

$$V_j\left(t_j, t_{-j} | \alpha_j, \beta_j, b_j\right) \geq V_j\left(t'_j, t_{-j} | \alpha_j, \beta_j, b_j\right),$$
$$\forall k \in \mathcal{J}, \forall c'_k \in C_k^{\mathcal{A}}, \forall t'_j \in T_j, \forall \left(\alpha_j, \beta_j\right) \in \Gamma_j\left(c'_k, c_{-k}\right) \times \Psi_j\left(M^{\mathcal{A}}\right),$$

and (ii) for all $\theta \in \Theta$,

$$U_i\left(\left(c'_j, c_{-j}\right), s, t, \theta\right) \geq U_i\left(\left(c'_j, c_{-j}\right), s', t, \theta\right), \forall j \in \mathcal{J}, \forall \left(c'_j, s'\right) \in C_j^{\mathcal{A}} \times S.$$

Given $c \in C^{\mathcal{A}}$ as the common prior belief on principals' equilibrium contracts, let $\mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi \rangle \text{-} C^{\mathcal{A}}}$ denote the set of all continuation equilibria when principals are restricted to offer contracts in $C^{\mathcal{A}}$.[11]

Definition 2 *(c, s, t) an equilibrium if (i) $(s, t) \in \mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi \rangle \text{-} C^{\mathcal{A}}}$, and (ii) for all $j \in \mathcal{J}$*

$$V_j\left(\left(c_j, c_{-j}\right), s, t\right) \geq V_j\left(\left(c'_j, c_{-j}\right), s, t\right), \forall c'_j \in C_j^{\mathcal{A}}.$$

[11] Later, we will allow principals to offer special (restricted) classes of contracts. For example, in the menu theorem in Peters (2001), principals are allowed to offer menus (i.e., a special class of contracts, which differs from $C^{\mathcal{A}}$).

3.5 Allocation

Let $Z \equiv \times_{j\in\mathcal{J}} \left[Z_j : \Theta \to Y_j\right]$ denote the set of allocations. For each strategy profile (c, s, t), define $z^{(c,s,t)} : \Theta \to Y$ as

$$z^{(c,s,t)}(\theta) = \left[z_j^{(c,s,t)}(\theta)\right]_{j\in\mathcal{J}} = \left[t_j\left(c_j, s_j(c,\theta)\right)\right]_{j\in\mathcal{J}}, \forall\theta \in \Theta.$$

I.e., $z^{(c,s,t)}$ is the allocation induced by (c, s, t). Let $Z^{\mathcal{E}\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}C^{\mathcal{A}}}$ be the set of all equilibrium allocations:

$$Z^{\mathcal{E}\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}C^{\mathcal{A}}} \equiv \left\{z^{(c,s,t)} \in Z : (c,s,t) \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}C^{\mathcal{A}}}\right\} \tag{6}$$

We aim to characterize $Z^{\mathcal{E}\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}C^{\mathcal{A}}}$. This is a difficult task, because $C^{\mathcal{A}}$ is complicated. When $\mathcal{A} = \mathcal{A}^{\text{delegated}}$, Peters (2001) achieves this goal by the menu theorem, which will be reviewed in Section 4. The goal of this paper is to characterize $Z^{\mathcal{E}\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}C^{\mathcal{A}}}$ when $\mathcal{A} = \mathcal{A}^{\text{non-delegated}}$.

4 The menu theorem in Peters (2001)

4.1 The menu theorem for delegated contracts

The menu theorem in Peters (2001) is established with delegated contracts, i.e., $\mathcal{A} = \mathcal{A}^{delegated}$. A menu introduced in Peters (2001) is a contract, $c_j : E_j \to Y_j$, such that

$$E_j \in 2^{Y_j} \diagdown \{\varnothing\} \text{ and } c_j\left(y_j\right) = y_j, \forall y_j \in E_j.$$

Let C_j^P denote the set of all menus for principal j, $C^P \equiv \times_{j\in\mathcal{J}} C_j^P$, and $C_{-j}^P \equiv \times_{k\neq j} C_k^P$. Let $M_j^P \equiv Y_j$ denote the set of messages used in all possible menus, $M^P \equiv \times_{j\in\mathcal{J}} M_j^P$, and $M_{-j}^P \equiv \times_{k\neq j} M_k^P$.

With slight abuse of notation, we continue to use the same notation of Γ, Ψ, (s,t) and $\mathcal{B}_j^{(c,s)}$ in the menu game where principals are restricted to offer contracts in C^P. In particular, every possible Γ and Ψ are defined by replacing $C_{-j}^{\mathcal{A}}$, $C^{\mathcal{A}}$, $M_{-j}^{\mathcal{A}}$ and $M^{\mathcal{A}}$ in (1) - (4) with C_{-j}^P, C^P, M_{-j}^P and M^P respectively. The agent's strategy is $s \equiv \left[s_j : C_j^P \times \Theta \to M_j^P\right]_{j\in\mathcal{J}}$, and let S^P denote the set of all such strategies. Each principal j's action choice strategy is a function $t_j : \Gamma_j\left(C^P\right) \times \Psi_j\left(M^P\right) \rightarrow Y_j$, and let T_j^P denote the set of all such action choice strategies. $\mathcal{B}_j^{(c,\,s)}$ is defined by replacing $C^{\mathcal{A}}$, M, and $C_j^{\mathcal{A}}$ in (5) with C^P, M^P, and C_j^P respectively.

In a model $\langle \mathcal{A},\, \Gamma,\, \Psi \rangle$, let $\mathcal{E}^{\langle c \,|\, \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}$ denote the set of all continuation equilibria conditional on given $c \in C^P$ as the common prior belief on principals' equilibrium contracts. A continuation equilibrium is defined by replacing $C^{\mathcal{A}}$, $C_k^{\mathcal{A}}$, and $C_j^{\mathcal{A}}$ in Definition 1 with C^P, C_k^P, and C_j^P respectively. Let $\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}$ be the set of all equilibria in the menu game where principals are restricted to offer contracts in C^P. An equilibrium is defined by replacing $C_j^{\mathcal{A}}$ in Definition 2 with C_j^P. Let $Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}}$ be the set of all equilibrium allocations defined by replacing $C^{\mathcal{A}}$ with C^P in (6).

Theorem 1 (The menu Theorem, Peters (2001)) *We have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^{\mathcal{A}}}} = Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}}, \tag{7}$$

$$\forall \langle \mathcal{A},\, \Gamma,\, \Psi \rangle \in \left\{\mathcal{A}^{delegated}\right\} \times \left\{\Gamma^{private}, \Gamma^{public}\right\} \times \left\{\Psi^{private}, \Psi^{public}\right\}.$$

With delegated contracts (i.e., $\mathcal{A} = \mathcal{A}^{delegated}$), the announcement and communication structures do not have impact on equilibiria[12].

The implication of Theorem 1 is that, given delegated contracts, it suffers no loss of generality for principals to offer menus on both the equilibrium path and the off-equilibrium paths. Since C^P is a much simpler set than $C^{\mathcal{A}}$, this result substantially simplifies the characterization of equilibrium allocations.

4.2 Failure of the menu theorem for non-delegated contracts

(7) in Theorem 1 can be dissected into two parts:

$$Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^{\mathcal{A}}}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}} \text{ and } Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^{\mathcal{A}}}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}C^P}}.$$

[12] Different announcement and communication structures lead to different information for principals *only after they offer their contracts*. With delegated contracts, principals do not make any strategic decision after offerring their contracts, and hence, the information (induced by different the announcement and communication structures) is irrelevant.

Let us modify the example in Section 2 by deleting agents 2 and 3. In this examples, principals offers menu contracts to agent 1, and for the equilibrium described in Section 2 (i.e. (T, H)), principal 1 achieves the max min value. However, it is straightfoward to see that, in any equilibrium in $Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}}$, principal 1 must achieve at least her min max value. This shows the failure of "$Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}} \supset Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$" regardless of $\langle \Gamma, \Psi\rangle$.

If "$Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}} \subset Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$" holds, we still have a weak sense of the menu theorem in that $Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$ serves as a superset of $Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}}$. With $\langle \Gamma^{private}, \Psi^{private}\rangle$, we will prove "$Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}} \subset Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$" in Secton 6 (i.e., Theorem 3).

In this section, we focus on $\langle \Gamma^{public}, \Psi^{public}\rangle$, $\langle \Gamma^{private}, \Psi^{public}\rangle$, and $\langle \Gamma^{public}, \Psi^{private}\rangle$, and we use examples to show failure of "$Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}} \subset Z^{\mathcal{E}\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$" in Sections 4.2.1, 4.2.2 and 4.2.3, respectively.

4.2.1 Public announcement and public communication

We construct an example with

$$\Theta = \{\theta, \theta'\}, \mathcal{I} = \{i\}, \mathcal{J} = \{j_1, j_2\}, Y_{j_1} = Y_{j_2} = \{1, 2, 3, 4\}.$$

The common prior is $\mu(\theta) = \mu(\theta') = 1/2$. The preference is defined as follows.

$$v_{j_1}\left[(y_{j_1}, y_{j_2}), \theta\right] = v_{j_2}\left[(y_{j_1}, y_{j_2}), \theta\right] = \begin{cases} 8, & \text{if } (y_{j_1}, y_{j_2}) = (1,1); \\ 1, & \text{if } (y_{j_1}, y_{j_2}) \neq (1,1) \text{ and } (y_{j_1} + y_{j_2}) \text{ is even;} \\ -1, & \text{if } (y_{j_1}, y_{j_2}) \neq (1,1) \text{ and } (y_{j_1} + y_{j_2}) \text{ is odd} \end{cases}$$

$$v_{j_1}\left[(y_{j_1}, y_{j_2}), \theta'\right] = v_{j_2}\left[(y_{j_1}, y_{j_2}), \theta'\right] = \begin{cases} 8, & \text{if } (y_{j_1}, y_{j_2}) = (4,4); \\ 1, & \text{if } (y_{j_1}, y_{j_2}) \neq (4,4) \text{ and } (y_{j_1} + y_{j_2}) \text{ is even;} \\ -1, & \text{if } (y_{j_1}, y_{j_2}) \neq (4,4) \text{ and } (y_{j_1} + y_{j_2}) \text{ is odd} \end{cases}$$

$$u_i\left[(y_{j_1}, y_{j_2}), \theta\right] = u_i\left[(y_{j_1}, y_{j_2}), \theta'\right] = \begin{cases} 1, & \text{if } (y_{j_1}, y_{j_2}) = (1,4); \\ 0, & \text{otherwise} \end{cases}$$

Consider the following equilibrium.

$$\text{on the equilibrium path: } \begin{pmatrix} \text{principal } j_1 \text{ offers a menu of two subsets: } \{1,2\} \text{ and } \{3,4\} \\ \text{principal } j_2 \text{ offers a menu of two subsets: } \{1,2\} \text{ and } \{3,4\} \end{pmatrix}$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{at state } \theta_1: \text{ agent } i \text{ chooses } \{1,2\} \text{ for both principals} \\ \text{at state } \theta_4: \text{ agent } i \text{ chooses } \{3,4\} \text{ for both principals} \end{pmatrix}$$

$$\text{on the equilibrium path: } \begin{pmatrix} \text{principals chooses } (1,1)\text{, if } i \text{ chooses } \{1,2\} \text{ for both principals;} \\ \text{principals chooses } (4,4)\text{, if } i \text{ chooses } \{3,4\} \text{ for both principals;} \\ \text{both principals chooses odd numbers, otherwise.} \end{pmatrix}$$

This is an equilibrium, because both principals achieve their maximal utility at both states. Therefore, principals do not have profitable deviations. Furthermore, the agent does not have a profitable deviation, because the agent achieves the minimal utility, regardless of his/her move.

Note that, for any unilateral deviation of any principal and any messages chosen by the agent, a continuation equilibrium always exists, because principals preference are fully aligned. Since we have public communication, principals observe the sets of actions available for both of them that are induced by agents' messages, they can choose $\left(y_{j_1}, y_{j_2}\right)$ with $\left(y_{j_1} + y_{j_2}\right)$ being even, which maximizes their utilities. Furthermore, for any message chosen by the agent, the agent always gets zero utility.

However, the equilibrium allocation described above cannot be replicated by menu contracts. We prove this by contradiction. Suppose that we can do it. Then, at state θ, we must get $(1,1)$, i.e., the unique best for principals, and at state θ', we must get $(4,4)$, i.e., the unique best for principals. Thus, both principals' equilibrium menu contract must include both 1 and 4. Then, $(1,1)$ cannot be achieved at state θ, because the agent would choose 1 from principal j_1's menu and 4 from principal j_2's menu, which achieves the maximal utility for the agent.

4.2.2 private announcement and public communication

The example above also works for private announcement and public communication. The only technical obstacle is the existence of continuation equilibrium when any principal deviates. We now show existence of continuation equilibrium holds.

For any $j \in \{j_1, j_2\}$, let the principal choose the following strategy whenever she sees off-equilibrium message.

$$\left[\begin{array}{c}\left(\begin{array}{c}\text{if the agent sends } m_j=\{1,2\} \text{ and } m_{-j}\notin\{\{1,2\},\{3,4\}\},\\ \text{principal } j \text{ choose } 2\end{array}\right),\\ \left(\begin{array}{c}\text{if the agent sends } m_j=\{3,4\} \text{ and } m_{-j}\notin\{\{1,2\},\{3,4\}\},\\ \text{principal } j \text{ choose } 3\end{array}\right)\end{array}\right]$$

To justify such a strategy, principal j needs to adopt the following beliefs:

$$\left(\begin{array}{c}\text{given } m_j=\{1,2\} \text{ and } m_{-j}\notin\{\{1,2\},\{3,4\}\}:\\ \text{principal } -j \text{ offers } c_{-j}:M_{-j}\to\mathcal{A}_{-j} \text{ with } c_{-j}\left(M_{-j}\right)=\{\{2\}\};\\ \\ \text{given } m_j=\{3,4\} \text{ and } m_{-j}\notin\{\{1,2\},\{3,4\}\}:\\ \text{principal } -j \text{ offers } c_{-j}:M_{-j}\to\mathcal{A}_{-j} \text{ with } c_{-j}\left(M_{-j}\right)=\{\{3\}\}\end{array}\right).$$

Given such beliefs, principal j's strategy is a best reply.

Furthermore, given principal j choosing either 2 or 3, the agent always gets 0 utility. Therefore, incentive compatibility of the agent holds.

Finally, given the agent's public communication, and principal j's strategy described above, the deviating principal $-j$ faces a finite set of options, and she can choose her best one. This completes the description of the continuation equilibrium when either principal $-j$ or the agent deviates.

4.2.3 public announcement and private communication

Now, we focus on public announcement and private communication with

$$\Theta=\{\theta\}, \text{and } |\Theta|=1, \mathcal{I}=\{i\}, \mathcal{J}=\{j_1,j_2\}, Y_{j_1}=\{-1,1\}, Y_{j_2}=\{0,1\},$$

i.e., it is a complete-information setup.

The preference is defined as follows.

$$v_{j_1}\left[\left(y_{j_1},y_{j_2}=0\right),\theta\right]=v_{j_2}\left[\left(y_{j_1},y_{j_2}=0\right),\theta\right]=u_i\left[\left(y_{j_1},y_{j_2}=0\right),\theta\right]=0, \forall y_{j_1}\in Y_{j_1},$$

$$v_{j_1}\left[\left(y_{j_1}=1,y_{j_2}=1\right),\theta\right]=v_{j_2}\left[\left(y_{j_1}=1,y_{j_2}=1\right),\theta\right]=u_i\left[\left(y_{j_1}=1,y_{j_2}=1\right),\theta\right]=1,$$

$$v_{j_1}\left[\left(y_{j_1}=-1,y_{j_2}=1\right),\theta\right]=u_i\left[\left(y_{j_1}=-1,y_{j_2}=1\right),\theta\right]=8,$$
$$v_{j_2}\left[\left(y_{j_1}=-1,y_{j_2}=1\right),\theta\right]=-8.$$

Consider the following equilibrium.

$$\text{on the equilibrium path: }\left(\begin{array}{c}\text{principal } j_1 \text{ offers a menu of one subset: } \{1\}\\ \text{principal } j_2 \text{ offers a menu of one (non-delegate) subset: } \{0,1\}\end{array}\right)$$

$$\text{on the equilibrium path: }\left(\begin{array}{c}\text{the agent chooses } \{1\} \text{ for principal } j_1,\\ \text{the agent chooses } \{0,1\} \text{ for principal } j_2\end{array}\right)$$

$$\text{on the equilibrium path: }\left(\begin{array}{c}\text{principal } j_1 \text{ chooses 1 from the subset } \{1\},\\ \text{principal } j_2 \text{ chooses 1 from the subset } \{0,1\},\end{array}\right)$$

On the equilibrium path, principal j_1 and the agent has degenerate decision, and hence their incentive compatibility holds. Also, incentive compatibility of principal j_2 holds, because she achieves the maximal utility given principal j_1 choosing 1.

Given principal j_1 choosing 1, principal j_2 achieves her maximal utility, and hence any deviation of principal j_2 is not profitable. But, we still need to show existence of continuation equilibrium. If principal j_2 deviates to $c_{j_2}: M_{j_2} \rightarrow 2^{Y_{j_2}} \diagdown \{\varnothing\}$ such that $c_{j_2}\left(M_{j_2}\right)=\{\{0\}\}$, the decisions of all principals and the agent are degenerate, and hence, incentive compatibility holds. Otherwise, there exists $m_{j_2} \in M_{j_2}$ such that $1 \in c_{j_2}\left(m_{j_2}\right)$. Then, the agent sends m_{j_2} to principal j_2, and j_2 chooses 1, which achieves the maximal utility for principal j_2 and the agent, given j_1 choosing 1. Therefore, incentive compatibility of j_2 and the agent holds, i.e., a continuation equilibrium exists.

Finally, principal j_1 does not have a profitable deviation either. To see this, we consider two cases: (1) j_1 deviates to $c_{j_1}: M_{j_1} \rightarrow 2^{Y_{j_1}} \diagdown \{\varnothing\}$ such that $c_{j_1}\left(M_{j_1}\right)=\{\{1\}\}$ — clearly, this is not a profitable deviation. Case (2), otherwise, i.e., there exists $m_{j_1} \in M_{j_1}$ such that $-1 \in c_{j_1}\left(m_{j_1}\right)$. In this case, we let the agent sends m_{j_1} to principal j_1, and j_1 chooses -1. Meanwhile, principal j_2 chooses 0 from the subset $\{0,1\}$. Given principal j_2 choosing 0, principal j_1 and the agent are indifferent among any choices of principal j_1, and hence, incentive compatibility of principal j_1 and the agent holds. Since principal j_1 would choose -1, it is a best reply for j_2 to choose 0 (rather than 1). Clearly, any deviation in case (2) is not profitable for principal j_1.

However, the equilibrium allocation described above cannot be replicated by menu contracts. We prove by contradiction. Suppose that we can use menu contracts to replicate it. In particular, on the equilibrium path, principals choose $\left(y_{j_1} = 1, y_{j_2} = 1\right)$. Thus, principal j_2's equilibrium menu contract must contain 1. Then, principal j_1 has a profitable deviation, i.e., deviates to the degenerate menu $\{-1\}$. Then, the agent would choose $\left(y_{j_1} = -1, y_{j_2} = 1\right)$, and achieve the highest utility, i.e., 8 for both principal j_1 and the agent. This is a profitable deviation for principal j_1.

5 Simpler contract spaces than $C^{\mathcal{A}}$

Given $\mathcal{A} = \mathcal{A}^{non-delegated}$, we follow the same strategy as Peters (2001) to characterize equilibrium allocations. Tha is, we identify two simple contract spaces, and prove that it suffers no loss of generality for principals to focus on these two simple contract spaces, one on the equilibrium path and the other off the equilibrium path. Given $\mathcal{A} = \mathcal{A}^{delegated}$, Peters (2001) shows that the two contract spaces are the same and they are C^P. We show that they are not the same.

5.1 Menu-of-menu-with-recommendation contracts

Given $j \in \mathcal{J}$, pick any $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$, and we say that $\left[E_j, y_j\right]$ is a menu with a recommendation if and only if $y_j \in E_j$. Define

$$M_j^R \equiv \left\{\left[E_j, y_j\right] : E_j \in 2^{Y_j} \diagdown \{\varnothing\} \text{ and } y_j \in E_j\right\},$$

i.e., M_j^R is the set of all menus with a recommendation. Let $M^R \equiv \times_{j \in \mathcal{J}} M_k^R$ and $M_{-j}^R \equiv \times_{k \neq j} M_k^R$.

For notational ease, define a function $\phi : M_j^R \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ as

$$\phi\left(\left[E_j, y_j\right]\right) = E_j, \forall \left[E_j, y_j\right] \in M_j^R.$$

That is, ϕ maps each $\left[E_j, y_j\right]$ to E_j.

Definition 3 *A menu-of-menu-with-recommendation contract for principal j is a pair of (i) $K_j \subset M_j^R$ and (ii) a function, $c_j : K_j \to 2^{Y_j} \diagdown \{\varnothing\}$ such that*

$$c_j\left(D_j\right) = \phi\left(D_j\right), \forall D_j \in K_j.$$

Note that when the agent sends a message $D_j = \left[E_j, y_j\right]$ in K_j, it is equivalent to choosing the menu of actions E_j along with recommending y_j to principal j. Nonetheless, this recommendation is not binding, so principal j can choose any action from E_j. For example, suppose that $K_j = \{[\{a,b\}, a], [\{c,d\}, c], [\{c,d\}, d], [\{e,f,g\}, e], [\{e,f,g\}, f]\}$. Then, a menu-of-menu-with-recommendation contract $c_j : K_j \to 2^{Y_j} \diagdown \{\varnothing\}$ has the following property:

$$c_j([\{a,b\}, a]) = \{a,b\}, c_j([\{c,d\}, c]) = c_j([\{c,d\}, d]) = \{c,d\},$$
$$c_j([\{e,f,g\}, e]) = c_j[\{e,f,g\}, f] = \{e,f,g\}$$

Let C_j^R be the set of all possible menu-of-menu-with-recommendation contracts for principal j, $C^R \equiv \times_{j \in \mathcal{J}} C_k^R$ and $C_{-j}^R \equiv \times_{k \neq j} C_k^R$.

Any menu (as defined in Section 4.1), e.g., $\{a,b,c\}$, can be viewed as a menu-of-menu-with-recommendation contract because we can set $K_j = \{[\{a\}, a], [\{b\}, b], [\{c\}, c]\}$. Therefore, C^P can be viewed as a strict subset of C^R.

5.2 Menu-of-menu-with-full-recommendation contracts

We now define another class of contracts. Pick any $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$, the following is a menu of E_j with *full recommendation*.

$$\left\{\left[E_j, y_j\right] : y_j \in E_j\right\}.$$

For example, if $E_j = \{a,b,c\}$, then a menu of $\{a,b,c\}$ with full recommendation is a set with three elements.

$$\{[\{a,b,c\}, a], [\{a,b,c\}, b], [\{a,b,c\}, c]\}.$$

Let N_j^F denote principal j's set of all such menus, i.e.,

$$N_j^F \equiv \left\{\left\{\left[E_j, y_j\right] : y_j \in E_j\right\} : E_j \in 2^{Y_j} \diagdown \{\varnothing\}\right\}.$$

Definition 4 *A menu-of-menu-with-full-recommendation contract for principal j is a pair of (i) $\left[L_j \subset 2^{Y_j} \text{ and } H_j \in N_j^F\right]$ and (ii) a function, $c_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ such that*

$$L_j \cap \phi\left(H_j\right) = \varnothing,$$

and

$$\begin{aligned} c_j\left(D_j\right) &= D_j, \forall D_j \in L_j, \\ c_j\left(\left[E_j, y_j\right]\right) &= E_j, \forall \left[E_j, y_j\right] \in H_j. \end{aligned}$$

Let C_j^F denote the set of all menu-of-menu-with-full-recommendation contracts for principal j, $C^F \equiv \times_{j \in \mathcal{J}} C_j^F$, and $C_{-j}^F \equiv \times_{k \neq j} C_k^F$. Let M_j^F denote the set of all messages that could possibly be included in the domain of a menu-of-menu-with-full-recommendation contracts for principal j, i.e.,

$$M_j^F = 2^{Y_j} \cup M_j^R.$$

Let $M^F \equiv \times_{j \in \mathcal{J}} M_j^F$, and $M_{-j}^F \equiv \times_{k \neq j} M_k^F$

Let us illustrate the domain of a menu-of-menu-with-full-recommendation contract, $c_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$. The set $L_j \subset 2^{Y_j}$ is a (possibly empty) subset of Y_j, and H_j is a menu of E_j-with-full recommendation such that E_j is non-empty and $E_j \notin L_j$. Suppose that $L_j \cup H_j$ is given as follows.

$$L_j \cup H_j = \{\{a, b\}, \{c, d, e\}, [\{f, g, h\}, f], [\{f, g, h\}, g], [\{f, g, h\}, h]\}.$$

Then, $c_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ satisfies the following properties:

$$\begin{aligned} c_j(\{a, b\}) &= \{a, b\}, c_j(\{c, d, e\}) = \{c, d, e\}, \\ c_j([\{f, g, h\}, f]) &= c_j([\{f, g, h\}, g]) = c_j([\{f, g, h\}, h]) = \{f, g, h\}. \end{aligned}$$

The interpretation of c_j is as follows. Principal j asks the agent to choose a subset in the menu of menus $\{\{a, b\}, \{c, d, e\}, \{f, g, h\}\}$. Therefore, the agent may choose $\{a, b\}$, or $\{c, d, e\}$, or $\{f, g, h\}$. And, if and only if the agent chooses $\{f, g, h\}$, the agent must recommend an action in $\{f, g, h\}$, i.e., f or g or h. Nevertheless, the recommendation is not binding. Therefore, all of the three choices, $[\{f, g, h\}, f]$, $[\{f, g, h\}, g]$, and $[\{f, g, h\}, h]$ that the agent chooses induce the same subset $\{f, g, h\}$ for principal j.

Any menu (as defined in Section 4.1), e.g., $\{a, b, c\}$, can be viewed as a menu-of-menu-with-full-recommendation contract because we can set $L_j = \{\{a\}, \{b\}\}$ and $H_j =$

$\{[\{c\},c]\}$ since c is one and only recommendation in $\{c\}$. Therefore, C^P is a strict subset of C^F. Furthermore, given any menu-of-menu-with-full-recommendation contract, $c_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$, since a recommendation is non-binding, for each $D_j \in L_j$, we can add an arbitrary recommendation $y_j \in D_j$. Thus, any menu-of-menu-with-full-recommendation contract can be viewed as a menu-of-menu-with-recommendation contract, i.e.,

$$C^P \subsetneq C^F \subsetneq C^R.$$

Our goal is to identify contract spaces that are as small as possible. It will be clear that we need C^R on the equilibrium path, and on off-equilibrium paths, it is enough to consider C^F.

5.3 $\left[C^I, C^{II}\right]$-equilibrium

Either $\mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{\mathcal{A}}}$ or $\mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi\rangle\text{-}C^{P}}$ is the set of equilibria where principals are restricted to use contracts from the same class both on and off the equilibrium path. We now formulate a notion of equilibrium where principals offer contracts from C^I on the equilibrium path and from C^{II} off the path for any $C^I, C^{II} \in \left\{C^{\mathcal{A}}, C^P, C^F, C^R\right\}$.

For any $I, II \in \{\mathcal{A}, P, F, R\}$ and any $j \in \mathcal{J}$, let $C_j^{I \cup II} \equiv C_j^I \cup C_j^{II}$, $C^{I \cup II} \equiv \times_{j \in \mathcal{J}} C_j^{I \cup II}$, and $C_{-j}^{I \cup II} \equiv \times_{k \neq j} C_k^{I \cup II}$. For any $I, II \in \{\mathcal{A}, P, F, R\}$ and any $j \in \mathcal{J}$, we also let $M_j^{I \cup II} \equiv M_j^I \cup M_j^{II}$, $M^{I \cup II} \equiv \times_{j \in \mathcal{J}} M_j^{I \cup II}$, and $M_{-j}^{I \cup II} \equiv \times_{k \neq j} M_k^{I \cup II}$. With slight abuse of notation, we continue to use the same notation of Γ, Ψ, (s,t) and $\mathcal{B}_j^{(c,\,s)}$. In particular, every possible Γ and Ψ are defined by replacing $C_{-j}^{\mathcal{A}}$, $C^{\mathcal{A}}$, $M_{-j}^{\mathcal{A}}$ and $M^{\mathcal{A}}$ in (1) - (4) with $C_{-j}^{I \cup II}$, $C^{I \cup II}$, $M_{-j}^{I \cup II}$ and $M^{I \cup II}$ respectively. The agent's strategy is $s \equiv \left[s_j : C_j^{I \cup II} \times \Theta \to M_j^{I \cup II}\right]_{j \in \mathcal{J}}$, and let $S^{I \cup II}$ denote the set of all such strategies. Each principal j's action choice strategy is a function $t_j : \Gamma_j\left(C^{I \cup II}\right) \times \Psi_j\left(M^{I \cup II}\right) \to Y_j$, and let $T_j^{I \cup II}$ denote the set of all such action choice strategies. $\mathcal{B}_j^{(c,\,s)}$ is defined by replacing $C^{\mathcal{A}}$, $M^{\mathcal{A}}$, and $C_j^{\mathcal{A}}$ in (5) with $C^{I \cup II}$, $M^{I \cup II}$, and $C_j^{I \cup II}$ respectively.

Definition 5 *Fix a model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$. For any $I, II \in \{\mathcal{A}, P, F, R\}$, $[s,t]$ is a $\left[C^I, C^{II}\right]$-continuation equilibrium given $c = (c_1, \ldots, c_J) \in C^I$ as the common prior belief on principals' equilibrium contracts if there exists*

$$b_j \in \mathcal{B}_j^{(c,\,s)}, \forall j \in \mathcal{J},$$

such that (i) for all $j \in \mathcal{J}$,

$$V_j\left(t_j, t_{-j}|\alpha_j, \beta_j, b_j\right) \geq V_j\left(t'_j, t_{-j}|\alpha_j, \beta_j, b_j\right),$$

$$\forall k \in \mathcal{J}, \forall c'_k \in \{c_k\} \cup C_k^{II}, \forall t'_j \in T_j^{I \cup II}, \forall \left(\alpha_j, \beta_j\right) \in \Gamma_j\left(c'_k, c_{-k}\right) \times \Psi_j\left(M_k^{I \cup II} \times M_{-k}^{I}\right),$$

and (ii) for all $\theta \in \Theta$,

$$U_i\left(\left(c'_j, c_{-j}\right), s, t, \theta\right) \geq U_i\left(\left(c'_j, c_{-j}\right), s', t, \theta\right), \forall \left(c'_j, s'\right) \in \left[\{c_j\} \cup C_j^{II}\right] \times S^{I \cup II}.$$

Given $c \in C^I$ as the common prior belief on principals' equilibrium contracts, let $\mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$ denote the set of all $\left[C^I, C^{II}\right]$-continuation equilibria.

Definition 6 *Fix a model $\langle \mathcal{A}, \Gamma, \Psi\rangle$. For any $I, II \in \{\mathcal{A}, P, F, R\}$, (c, s, t) an $\left[C^I, C^{II}\right]$-equilibrium if (i) $c = (c_1, \ldots, c_J) \in C^I$ (ii) $(s, t) \in \mathcal{E}^{\langle c \mid \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$, and (ii) for all $j \in \mathcal{J}$*

$$V_j\left(\left(c_j, c_{-j}\right), s, t\right) \geq V_j\left(\left(c'_j, c_{-j}\right), s, t\right), \forall c'_j \in C_j^{II}.$$

That is, in a $\left[C^I, C^{II}\right]$-equilibrium (c, s, t), principals offer $c \in C^I$ on the equilibrium path, and it is not profitable for each principal j to deviate unilaterally to any $c'_j \in C_j^{II}$. Furthermore, (s, t) is a $\left[C^I, C^{II}\right]$-continuation equilibrium given $c \in C^I$ as the common prior belief on the equilibrium contracts.

In a model $\langle \mathcal{A}, \Gamma, \Psi\rangle$, let $\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}$ denote the set of all $\left[C^I, C^{II}\right]$-equilibria, and

$$Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}} \equiv \left\{z^{(c,s,t)} : (c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^I, C^{II}\right]}\right\}.$$

In a non-delegated model regardless of the announcement and communication structure, we will show that it suffers no loss of generality for principals to offer contracts in C^R, and on off-equilibrium paths to offer contracts in C^F. If both annoucement and communication are private, C^R on the equilibrium path can be replaced with $C^P \subsetneq C^R$.

6 Private announcement and private communication

Given $\mathcal{A} = \mathcal{A}^{non-delegated}$, we first consider the model of private announcement and private communication. In this model, there is no loss of generality to focus on $\left[C^P, C^F\right]$-equilibria.

Theorem 2 *In the model* $\langle \mathcal{A},\, \Gamma,\, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\, \Gamma^{private},\, \Psi^{private} \right\rangle$*, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{P}, C^{F}\right]}}.$$

Theorem 2 shows that the menu theorem in Peters (2001) (Theorem 1) holds only on the equilibrium path in the model $\langle \mathcal{A},\, \Gamma,\, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\, \Gamma^{private},\, \Psi^{private} \right\rangle$. However, it is not enough to use them off the equilibrium path. In particular, an equilibrium $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{P}, C^{P}\right]}$ may not survive when a principal can deviate to a more complex mechanism. The example in Szentes (2009) shows that. Upon principal 1's deviation to a contract that leaves the whole set of $\{T, H\}$ with him regardless of the agent's message, the unique continuation equilibrium is that principal 1's action choice and the agent's recommendation to principal 2 are the same. In this continuation equilibrium, principal 1's utility is one which is strictly greater than her max min value. What Theorem 2 shows that a principal can only consider menu-of-menu-with-full-recommendation contracts for her deviation without loss of generality.

Theorem 2 is proved by two steps. First, Proposition 1 shows that there is no loss of generality to focus on menu contracts as equilibrium contracts regardless of the complexity of contracts allowed on and off the equilibrium paths. Second, Proposition 2 shows that there is no loss of generality to focus on menu-of-menu-with-full-recommendation contracts for possible contracts that a deviating principal may choose even if she can choose any complex mechanisms. The proof of Proposition 1 is relegated to Appendix, whereas the proof of Proposition 2 is found in Section 6.1.

Proposition 1 *In the model* $\langle \mathcal{A},\, \Gamma,\, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\, \Gamma^{private},\, \Psi^{private} \right\rangle$*, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{P}, C^{\mathcal{A}}\right]}}.$$

For the proof of Proposition 1, we first start with $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$. We can then construct a menu contract $\bar{c}_j : E_j \to Y_j$ for principal j with the domain $E_j \equiv \left\{ t_j \left(c_j \left(m_j \right) \right) : m_j \in M_j^{\mathcal{A}} \right\}$, that is, principal j delegates any action she could have chosen in c_j to the agent. Subsequently, we can construct a continuation equilibrium $\left[C^{P}, C^{\mathcal{A}}\right]$-continuation equilibrium $\left[\bar{s}, \bar{t}\right]$ given $\bar{c} = \left[\bar{c}_j\right]_{j \in \mathcal{J}} \in C^{P}$ as the prior belief on equilibrium contract such that an equilibrium $\left(\bar{c}, \bar{s}, \bar{t}\right)$ in $\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle\text{-}\left[C^{P}, C^{\mathcal{A}}\right]}$ with $z^{\left[\bar{c},\, \bar{s},\, \bar{t}\right]} = z^{(c,s,t)}$.

On the other hand, consider $(c,s,t) \in \mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^{\mathcal{A}}]}$. The key step for the proof of the other way around is how to construct an equilibrium contract $\bar{c}_j \in C_j^{\mathcal{A}}$. Given the domain E_j of equilibrium menu contract c_j, we chooses an injective function $\psi_j : E_j \longrightarrow M_j^{\mathcal{A}}$ so that each $\psi_j(y_j)$ fully represents y_j for each $y_j \in E_j$, . We can then relabel each $y_j \in E_j$ to $\psi_j(y_j)$, replicating c_j to $\bar{c}_j|_{\psi_j(E_j)} : \psi_j(E_j) \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ such that $\bar{c}_j\left[\psi_j(y_j)\right] = y_j$ for all $y_j \in E_j$. Similarly, we replicate (s,t) to $(\bar{s},\bar{t})$ subject to such relabeling. As the final step, we extend each $\bar{c}_j|_{\psi_j(E_j)}$ to $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ as follows: Given an arbitrary $y_j' \in E_j$,

$$\bar{c}_j\left[\psi_j(y_j)\right] = y_j, \forall y_j \in E_j,$$
$$\bar{c}_j[m_j] = y_j', \forall m_j \in M_j^{\mathcal{A}} \diagdown \psi_j(E_j),$$

We can subsequently construct a $[C^{\mathcal{A}}, C^{\mathcal{A}}]$-continuation equilibrium $(\bar{s},\bar{t})$ such that $z^{[\bar{c},\bar{s},\bar{t}]} = z^{(c,s,t)}$.

Proposition 2 *In the model* $\langle \mathcal{A}, \Gamma, \Psi\rangle = \langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private}\rangle$, *we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^{\mathcal{A}}]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^F]}}.$$

Proposition 2 concerns what contracts a principal can focus on when she deviates, showing that contracts in C^F are sufficient for a principal's deviations without loss of generality. Because of Proposition 1, we focus on $Z^{\mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^{\mathcal{A}}]}}$. To show $Z^{\mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^{\mathcal{A}}]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^F]}}$, we start with an equilibrium $(c,s,t) \in \mathcal{E}^{\langle \mathcal{A},\Gamma,\Psi\rangle\text{-}[C^P,C^{\mathcal{A}}]}$ as shown in the proof in Section 6.1.1. We need to construct a $[C^P, C^F]$-continuation equilibrium $(\bar{s},\bar{t})$ that prevents a principal from deviating to any menu-of-menu-with-full recommendation contract $\widehat{c}_j' \in C_j^F$. For example, consider principal j's deviation to $\widehat{c}_j' : L_j \cup H_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ with

$$L_j \cup H_j = \{\{a,b\}, \{c,d,e\}, [\{f,g\},f], [\{f,g\},g]\}.$$

Then, $\widehat{c}_j' : L_j \cup H_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ satisfies the following properties:

$$\widehat{c}_j'(\{a,b\}) = \{a,b\}, \widehat{c}_j'(\{c,d,e\}) = \{c,d,e\},$$
$$\widehat{c}_j'([\{f,g\},f]) = \widehat{c}_j'([\{f,g\},g]) = \{f,g\}.$$

It is clear that there exists a contract $c'_j : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}$ such that

$$\begin{aligned} c'_j(m'_j) &= \{a, b\}, c'_j(m''_j) = \{c, d, e\}, \\ c'_j(m_j) &= \{f, g\}, \forall m_j \in M_j^{\mathcal{A}} \diagdown \{m'_j, m''_j\}. \end{aligned}$$

When principal j receives m'_j or m''_j, she chooses an action from $\{a, b\}$ or an action from $\{c, d, e\}$ respectively. We convert m'_j to $\{a, b\}$ in $\widehat{c}'_j$ and m''_j to $\{c, d, e\}$. There are potentially multiple messages in that principal j can receive in a continuation equilibrium, inducing $\{f, g\}$ and each of them leading to her different action choice from $\{f, g\}$. Messages that lead to f as principal j's action choice can be converted to $[\{f, g\}, f]$ and messages that lead to g can be converted to $[\{f, g\}, g]$. In this way, we can preserve the agent's communication behavior that leads to principal j's same action choice. After all we can construct a continuation equilibrium in which principal j does not gain upon $\widehat{c}'_j$ because principal j does not gain in the original continuation equilibrium upon deviation to c'_j.

On the other hand, we fix $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^P, C^F]}$ to show the converse. We need to construct a $[C^P, C^{\mathcal{A}}]$-continuation equilibrium $(\bar{s}, \bar{t})$ that prevents a principal from deviating to any contract $\left[c'_j : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}\right] \in C_j^{\mathcal{A}}$. Because $M_j^{\mathcal{A}}$ has a much larger cardinality than $2^{Y_j} \diagdown \{\varnothing\}$, there exists $E_j^* \in c'_j\left(M_j^{\mathcal{A}}\right)$ such that

$$\left|\left\{m_j \in M_j^{\mathcal{A}} : c'_j(m_j) = E_j^*\right\}\right| > \left|E_j^*\right|.$$

Then, we pick $\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ with $L_j = c'_j\left(M_j^{\mathcal{A}}\right) \diagdown \left\{E_j^*\right\}$ and $H_j = \left\{\left[E_j^*, y_j\right] : y_j \in E_j^*\right\}$. As the proof in Section 6.1.2 shows, we can construct a $[C^P, C^{\mathcal{A}}]$-continuation equilibrium $(\bar{s}, \bar{t})$ in which principal j does not gain upon her deviation to c'_j, from the $[C^P, C^F]$-continuation equilibrium (s, t) upon her deviation to such a corresponding menu-of-menu-with-full-recommendation contract $\widehat{c}'_j$.

The following theorem shows that any equilibrium allocation in $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^P, C^F]}}$ belongs to $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^P, C^P]}}$ in the model $\langle \mathcal{A}, \Gamma, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private} \right\rangle$, the set of allocations of equilibria where a principal can choose only menus on and off the path.

Theorem 3 *In the model* $\langle \mathcal{A}, \Gamma, \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private} \right\rangle$, *we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^P, C^F]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^P, C^P]}}.$$

Proof. Fix $\langle \mathcal{A}, \Gamma, \Psi \rangle = \langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private} \rangle$. Pick any $(c,s,t) \in Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]}$. We replicate (s,t) with $\left(\bar{s},\bar{t}\right)$ only over (i) $c \in C^P$ and (ii) any principal j's unilateral deviation to any contract in C_j^P. Then, it is clear that $\left(\bar{s},\bar{t}\right) \in \mathcal{E}^{\langle c | \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]$ because $(s,t) \in Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]}$ and $C^P \subset C^F$. Further, we have that

$$V_j\left(\left(c_j, c_{-j}\right), s, t\right) \geq V_j\left(\left(c_j', c_{-j}\right), s, t\right) \; \forall c_j' \in C_j^F$$

Because (s,t) is replicated by $\left(\bar{s},\bar{t}\right) \in \mathcal{E}^{\langle c | \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]$ and $C^P \subset C^F$, the inequality above implies that

$$V_j\left(\left(c_j, c_{-j}\right), \bar{s}, \bar{t}\right) \geq V_j\left(\left(c_j', c_{-j}\right), \bar{s}, \bar{t}\right) \; \forall c_j' \in C_j^P,$$

which implies $\left(c,\bar{s},\bar{t}\right) \in Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$. ■

Because $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]} = Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$ in the model $\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private} \rangle$, Theorem 3 implies that $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$.[13] Of course, this does not mean that $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$. The example in Szentes (2009) can be used as a counter example. Therefore, we can conclude that every allocation in $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$ belongs to $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$ but not every allocation in $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$ belongs to $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$.

6.1 Proof of Proposition 2

6.1.1 Proof of $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^{\mathcal{A}}\right]} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]}$

In the model $\langle \mathcal{A}, \Gamma, \Psi \rangle = \langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{private} \rangle$, we first show $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^{\mathcal{A}}\right]} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]}$. Fix any $(c,s,t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^{\mathcal{A}}\right]$, and we aim to construct an equilibrium $\left(c, \bar{s}, \bar{t}\right) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^F\right]$ such that $z^{\left(c, \bar{s}, \bar{t}\right)} = z^{(c,s,t)}$.

[13]We can directly show this relation. Because we have that $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]} = Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^{\mathcal{A}}\right]}$ thanks to Proposition 1, we only need to show that $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^{\mathcal{A}}\right]} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle}\text{-}\left[C^P, C^P\right]}$. This is straightforward. Suppose that principal j deviates to a menu contract $c_j' : E_j \to Y_j$ in C_j^P. Note that there exists a corresponding contract $\tilde{c}_j' : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}$ in $C_j^{\mathcal{A}}$ such that each image is a singleton and the image set $\tilde{c}_j'\left(M_j^{\mathcal{A}}\right)$ is the same as E_j. Because it is not profitable to deviate to such a contract in $\tilde{c}_j' \in C_j^{\mathcal{A}}$, we can construct a continuation equilibrium upon the principal's deviation to the menu contract c_j' in which he does not gain.

First, let $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$ be the domain of c_j for each $j \in \mathcal{J}$. Define

$$\bar{s}(c,\theta) = s(c,\theta), \forall \theta \in \Theta,$$

$$\begin{aligned} \bar{t}_j(c_j, y_j) &= t_j(c_j, y_j) = y_j, \forall j \in \mathcal{J}, \forall y_j \in E_j, \\ \bar{b}_j(c_j, y_j) &= b_j(c_j, y_j), \forall j \in \mathcal{J}, \forall y_j \in E_j. \end{aligned}$$

i.e., if no principal deviates from c, we let $(\bar{s}, \bar{t})$ be equal to (s, t) with the same belief $\bar{b}_j(c_j, m_j)$ as $b_j(c_j, m_j)$. The $\bar{b}_j(c_j, m_j)$ clearly satisfies the requirement regarding Bayes' rule because $b_j(c_j, m_j)$ satisfies it, but it does not have a role in principal j's action choice since it is not strategic given her menu contract c_j. Subsequently, $z^{[c, \bar{s}, \bar{t}]} = z^{(c,s,t)}$.

Second, for any deviation of $\widehat{c}'_j \in C_j^F$ by principal j, we must construct a continuation equilibrium such that this deviation is not profitable for j. Since $\widehat{c}'_j \in C_j^F$, we have a pair of (i) $\left[L_j \subset 2^{Y_j} \text{ and } H_j = \{[E_j, y_j] : y_j \in E_j\}\right]$ and (ii) a mapping, $\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ such that $\phi(H_j) \notin L_j$ and

$$\begin{aligned} \widehat{c}'_j(D_j) &= D_j, \forall D_j \in L_j, \\ \widehat{c}'_j([E_j, y_j]) &= E_j, \forall [E_j, y_j] \in H_j. \end{aligned}$$

Fix an injective function $\varphi_j : L_j \longrightarrow M_j^{\mathcal{A}}$. That is, for each $D_j \in L_j$, the message $\varphi_j(D_j) \in M_j^{\mathcal{A}}$ fully represents D_j. We now define a contract $c'_j : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}$ as follows.

$$\begin{aligned} c'_j\left[\varphi_j(D_j)\right] &= D_j, \forall D_j \in L_j, \\ c'_j(m_j) &= E_j, \forall m_j \in M_j^{\mathcal{A}} \diagdown \varphi_j(L_j). \end{aligned}$$

That is, for every $D_j \in L_j$, the message $\varphi_j(D_j)$ pins down the subset $D_j \subset Y_j$, and any other messages (i.e., $m_j \in M_j^{\mathcal{A}} \diagdown \varphi_j(L_j)$) pins down the subset $E_j \subset Y_j$.

Clearly, $c'_j \in C_j^{\mathcal{A}}$. Since $(c,s,t) \in (c,s,t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} [C^P, C^{\mathcal{A}}]}$, c'_j is not a profitable deviation under (c,s,t). Based on this, we construct $(\bar{s}, \bar{t})$ as follows.

$$\begin{pmatrix} s_j\left(\left(c'_j, c_{-j}\right), \theta\right) = \varphi_j(D_j), \\ D_j \in L_j \end{pmatrix} \Longrightarrow \bar{s}_j\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right) = D_j, \forall \theta \in \Theta, \tag{8}$$

$$\begin{pmatrix} s_j\left(\left(c'_j, c_{-j}\right), \theta\right) \in M_j^{\mathcal{A}} \diagdown \varphi_j(L_j), \\ t_j\left[c'_j, s_j\left(\left(c'_j, c_{-j}\right), \theta\right)\right] = y_j \in E_j \end{pmatrix} \Longrightarrow \bar{s}_j\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right) = [E_j, y_j], \forall \theta \in \Theta, \tag{9}$$

$$\bar{s}_{-j}\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right) = s_{-j}\left(\left(c_j', c_{-j}\right), \theta\right), \forall\theta \in \Theta \tag{10}$$

(8) shows that if the agent's message pins down $D_j \in L_j$ under $\left(c_j', c_{-j}\right)$, we let the agent send the corresponding message that pins $D_j \in L_j$ under $\left(\widehat{c}_j', c_{-j}\right)$. (9) shows that if the agent's message pins down E_j under $\left(c_j', c_{-j}\right)$ and principal j subsequently chooses $y_j \in E_j$ under t_j, we let the agent send the corresponding message that pins $\left[E_j, y_j\right] \in H_j$ under $\left(\widehat{c}_j', c_{-j}\right)$.

Upon deviation to $\widehat{c}_j' \in C_j^F$, $\bar{t}_j$ and $\bar{b}_j$ replicate t_j and b_j respectively. First of all,

$$\bar{t}_j\left(\widehat{c}_j', D_j\right) = t_j\left(c_j', \varphi_j\left(D_j\right)\right), \forall D_j \in L_j, \tag{11}$$

$$\bar{b}_j\left(\widehat{c}_j', D_j\right) = b_j\left(c_j', \varphi_j\left(D_j\right)\right), \forall D_j \in L_j. \tag{12}$$

i.e., when the agent's message pins down $D_j \in L_j$, principal j chooses the same action under $\bar{t}_j$ given the same belief as b_j as she would do under t_j given b_j. The belief $\bar{b}_j$ in (12) satisfies the requirement regarding Bayes' rule as b_j satisfies it.

Secondly, by (9), the set of messages under $\bar{s}_j$ that could be sent from some of the agent's types to principal j is defined as follows.

$$Q_j = \left\{ \left[E_j, y_j\right] : \begin{array}{c} s_j\left(\left(c_j', c_{-j}\right), \theta\right) \in M_j^A \diagdown \varphi_j\left(L_j\right), \\ t_j\left[c_j', s_j\left(\left(c_j', c_{-j}\right), \theta\right)\right] = y_j \in E_j \end{array} \right\}.$$

Note that every element in Q_j has E_j in it. We define $\bar{t}_j$ and $\bar{b}_j$ when the agent sends a message $\left[E_j, y_j\right] \in Q_j$ to principal j upon j's deviation to $\widehat{c}_j'$ as follows.

$$\bar{t}_j\left(\widehat{c}_j', \left[E_j, y_j\right]\right) = t_j\left[c_j', s_j\left(\left(c_j', c_{-j}\right), \theta\right)\right], \forall\left[E_j, y_j\right] \in Q_j, \tag{13}$$

given the belief

$$\bar{b}_j\left(\widehat{c}_j', \left[E_j, y_j\right]\right) = b_j\left[c_j', s_j\left(\left(c_j', c_{-j}\right), \theta\right)\right], \forall\left[E_j, y_j\right] \in Q_j. \tag{14}$$

$\bar{b}_j$ in (14) satisfies the requirement regarding Bayes' rule since b_j satisfies it. (9) says that, if the agent's message pins down E_j under $\left(c_j', c_{-j}\right)$ and principal j subsequently chooses $y_j \in E_j$ under t_j, we let the agent send the corresponding message that pins $\left[E_j, y_j\right] \in H_j$ under $\left(\widehat{c}_j', c_{-j}\right)$. (13) says that principal j indeed follows the agent's recommendation of y_j given the same belief as b_j because $\left[E_j, y_j\right]$ is in Q_j and y_j is the action she would choose under b_j, i.e., $t_j\left[c_j', s_j\left(\left(c_j', c_{-j}\right), \theta\right)\right] = y_j \in E_j$. Thus, $z^{\left[\left(\widehat{c}_j', c_{-j}\right), \bar{s}, \bar{t}\right]} = z^{\left(\left(c_j', c_{-j}\right), s, t\right)}$.

Finally, we define $\bar{t}_j$ and $\bar{b}_j$ when the agent sends the message profile $\left[E_j, y_j\right] \in H_j \backslash Q_j$ upon j's deviation to $\widehat{c}'_j$. For all $\left[E_j, y_j\right] \in H$, let $m_j^* \in M_j^{\mathcal{A}}$ be an arbitrary message such that $c'_j\left(m_j^*\right) = E_j$.

$$\bar{t}_j\left(\widehat{c}'_j, \left[E_j, y_j\right]\right) = t_j\left(c'_j, m_j^*\right), \forall \left[E_j, y_j\right] \in H_j \backslash Q_j \tag{15}$$

given the belief

$$\bar{b}_j\left(\widehat{c}'_j, \left[E_j, y_j\right]\right) = b_j\left(c'_j, m_j^*\right), \forall \left[E_j, y_j\right] \in H_j \backslash Q_j \tag{16}$$

Suppose that the agent sends a message $\left[E_j, y_j\right] \in H_j \backslash Q_j$. We pick an arbitrary message $m_j^* \in M_j^{\mathcal{A}}$ such that $c'_j\left(m_j^*\right) = E_j$. Then, we replicate t_j and b_j associated with m_j^* with $\bar{t}_j$ and $\bar{b}_j$ associated with $\left[E_j, y_j\right] \in H_j \backslash Q_j$. The belief $\bar{b}_j$ in (16) satisfies the requirement regarding Bayes' rule because b_j satisfies it.

Because $\left(\left(c'_j, c_{-j}\right), s, t\right)$ specifies a continuation equilibrium upon j's deviation to c'_j given c_{-j}, our construction of strategy profiles and beliefs shows (i) $\left(\left(\widehat{c}'_j, c_{-j}\right), \bar{s}, \bar{t}\right)$ also specifies a continuation equilibrium upon j's deviation to $\widehat{c}'_j$ given c_{-j} and (ii) $z^{\left[\left(\widehat{c}'_j, c_{-j}\right), \bar{s}, \bar{t}\right]} = z^{\left(\left(c'_j, c_{-j}\right), s, t\right)}$. Therefore, deviation to $\widehat{c}'_j$ is not profitable. Thus, $\left[c, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}$ and $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$.

6.1.2 Proof of $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^F\right]}}$

We now show $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^F\right]}}$. Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^F\right]}$, and we aim to construct an equilibrium $\left[c, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}$ such that $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$.

First, let $E_j \in 2^{Y_j} \diagdown \{\varnothing\}$ be the domain of c_j for each $j \in \mathcal{J}$. Define

$$\bar{s}(c, \theta) = s(c, \theta), \forall \theta \in \Theta,$$

$$\begin{aligned} \bar{t}_j\left(c_j, y_j\right) &= t_j\left(c_j, y_j\right) = y_j, \forall j \in \mathcal{J}, \forall y_j \in E_j, \\ \bar{b}_j\left(c_j, y_j\right) &= b_j\left(c_j, y_j\right), \forall j \in \mathcal{J}, \forall y_j \in E_j. \end{aligned}$$

i.e., if no principal deviates from c, we let $\left[\bar{s}, \bar{t}\right]$ be equal to $[s, t]$. The belief $\bar{b}_j\left(c_j, m_j\right)$ satisfies the requirement regarding Bayes' rule because $b_j\left(c_j, m_j\right)$ satisfies it. As a result, $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$, the agent's and principals' incentive compatibility is preserved when no principal deviates from c.

Second, for any deviation of $c'_j \in C_j^{\mathcal{A}}$ by principal j, we must construct a continuation equilibrium such that this deviation is not profitable for j. Consider

$$\mathcal{L}_j \equiv \left\{ c'_j\left(m_j\right) \in 2^{Y_j} \diagdown \{\varnothing\} : m_j \in M_j^{\mathcal{A}} \right\}.$$

Since $M_j^{\mathcal{A}}$ has a much larger cardinality than $2^{Y_j} \diagdown \{\varnothing\}$, there exists $E_j^* \in \mathcal{L}_j$ such that

$$\left| \left\{ m_j \in M_j^{\mathcal{A}} : c'_j\left(m_j\right) = E_j^* \right\} \right| > \left| E_j^* \right|.$$

Consider $\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ defined as follows.

$$\begin{aligned} L_j &= \mathcal{L}_j \diagdown \left\{ E_j^* \right\}, \\ H_j &= \left\{ \left[E_j^*, y_j \right] : y_j \in E_j^* \right\}. \end{aligned}$$

and

$$\begin{aligned} \widehat{c}'_j\left(D_j\right) &= D_j, \forall D_j \in L_j, \\ \widehat{c}'_j\left(\left[E_j^*, y_j\right]\right) &= E_j^*, \forall \left[E_j^*, y_j\right] \in H_j. \end{aligned}$$

Clearly, $\widehat{c}'_j \in C_j^F$. Since $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} [C^P, C^F]}$, $\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)$ is a continuation equilibrium, and it is not profitable for j.

Fix any injective function $\varphi_j : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}$, such that

$$\begin{aligned} c'_j\left[\varphi_j\left(D_j\right)\right] &= D_j, \forall D_j \in L_j, \\ c'_j\left[\varphi_j\left(\left[E_j^*, y_j\right]\right)\right] &= E_j^*, \forall \left[E_j^*, y_j\right] \in H_j. \end{aligned}$$

That is, for each $D_j \in L_j$, we pick a message $\varphi_j\left(D_j\right) \in M_j^{\mathcal{A}}$ such that $\varphi_j\left(D_j\right)$ pins down D_j under c'_j. Furthermore, for each $\left[E_j^*, y_j\right] \in H_j$, we pick a distinct message $\varphi_j\left(\left[E_j^*, y_j\right]\right) \in M_j^{\mathcal{A}}$ such that $\varphi_j\left(\left[E_j^*, y_j\right]\right)$ pins down E_j^* under c'_j.

Hypothetically, let us delete messages in $M_j^{\mathcal{A}} \diagdown \varphi_j\left(L_j \cup H_j\right)$, i.e., we consider the contract c'_j with the restricted domain of $\varphi_j\left(L_j \cup H_j\right)$, which is denoted by $c'_j|_{\varphi_j\left(L_j \cup H_j\right)}$. As a result, the contract $\widehat{c}'_j$ is equivalent to the contract $c'_j|_{\varphi_j\left(L_j \cup H_j\right)}$ subject to re-labeling of messages via the bijection φ_j, which maps elements in $L_j \cup H_j$ to elements in $\varphi_j\left(L_j \cup H_j\right)$. Thus, we can define $\left(\left(c'_j|_{\varphi_j\left(L_j \cup H_j\right)}, c_{-j}\right), \overline{s}, \overline{t}\right)$ to replicate $\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)$, and $\left(\left(c'_j|_{\varphi_j\left(L_j \cup H_j\right)}, c_{-j}\right), \overline{s}, \overline{t}\right.$

is a continuation equilibrium, which inherits incentive compatibility of the agent and the principals under $\left(\left(\widehat{c}_j', c_{-j}\right), s, t\right)$. Rigorously, we define $\left(\bar{s}, \bar{t}\right)$ as follows.

$$\begin{aligned} s_j\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right) &= x_j \Longrightarrow \bar{s}_j\left(\left(c_j', c_{-j}\right), \theta\right) = \varphi_j\left(x_j\right), \forall x_j \in L_j \cup H_j, \forall \theta \in \Theta, \\ \bar{s}_{-j}\left(\left(c_j', c_{-j}\right), \theta\right) &= s_{-j}\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right), \forall \theta \in \Theta, \end{aligned}$$

and

$$\begin{aligned} \bar{t}_j\left(c_j', \varphi_j(x_j)\right) = t_j\left[\widehat{c}_j', x_j\right], \forall x_j \in L_j \cup H_j, \\ \bar{b}_j\left(c_j', \varphi_j(x_j)\right) = b_j\left[\widehat{c}_j', x_j\right], \forall x \in L_j \cup H_j. \end{aligned}$$

Note that the belief $\bar{b}_j\left(c_j', \varphi_j(x)\right)$ satisfies the requirement regarding Bayes' rule because $b_j\left[\widehat{c}_j', x_j\right]$ satisfies it. Finally, we extend the continuation $\left(\left(c_j'|_{\varphi_j(L_j \cup H_j)}, c_{-j}\right), \bar{s}, \bar{t}\right)$ to a continuation equilibrium $\left(\left(c_j', c_{-j}\right), \bar{s}, \bar{t}\right)$, i.e., we have to extend each principal j's strategy when the agent sends a message $m_j \notin \varphi_j\left(L_j \cup H_j\right)$. Fix any $\left[E_j^*, y_j^\circ\right] \in H_j$. Define

$$\begin{aligned} c_j'\left(m_j\right) &= x_j \in K_j \Longrightarrow \\ \bar{t}_j\left(c_j', m_j\right) &= t_j\left[\widehat{c}_j', x_j\right] = \bar{t}_j\left(c_j', \varphi_j\left(x_j\right)\right), \\ \bar{b}_j\left(c_j', m_j\right) &= b_j\left[\widehat{c}_j', x_j\right] = \bar{b}_j\left(c_j', \varphi_j\left(x_j\right)\right). \end{aligned}$$

The requirement regarding Bayes' rule is satisfied with $\bar{b}_j\left(c_j', m_j\right)$. In the situation where the agent's message m_j is not in $\varphi_j\left(L_j \cup H_j\right)$, if m_j pins down $x_j \in L_j$ under c_j', principal j associates m_j with $\varphi_j\left(x_j\right)$, or equivalently, upon receiving a message that pins down $x_j \in L_j$ under c_j', principal j forms the same belief as the one induced by receiving $\varphi_j\left(x\right)$, and hence, principal j takes the same best reply $\bar{t}_j\left(c_j', \varphi_j\left(x_j\right)\right)$.

$$\begin{aligned} c_j'\left(m_j\right) &= E_j^* \Longrightarrow \\ \bar{t}_j\left(c_j', m_j\right) &= t_j\left[\widehat{c}_j', \left[E_j^*, y_j^\circ\right]\right] = \bar{t}_j\left(c_j', \varphi_j\left(\left[E_j^*, y_j^\circ\right]\right)\right), \\ \bar{b}_j\left(c_j', m_j\right) &= b_j\left[\widehat{c}_j', \left[E_j^*, y_j^\circ\right]\right] = \bar{b}_j\left(c_j', \varphi_j\left(\left[E_j^*, y_j^\circ\right]\right)\right). \end{aligned}$$

The requirement regarding Bayes' rule is satisfied with $\bar{b}_j\left(c_j', m_j\right)$. In the situation where the agent's message m_j is not in $\varphi_j\left(L_j \cup H_j\right)$, if m_j pins down E_j^* under c_j', principal j associates m_j with $\varphi_j\left(\left[E_j^*, y_j^\circ\right]\right)$, or equivalently, upon receiving a message that pins down E_j^*, principal j forms the same belief as the one induced by receiving $\varphi_j\left(\left[E_j^*, y_j^\circ\right]\right)$, and hence, principal j takes the same best reply $\bar{t}_j\left(c'_j, \varphi_j\left(\left[E^*_j, y^\circ_j\right]\right)\right)$. As a result, these extra messages bring no harm to the agent's incentive compatibility and principals' incentive compatibility, and therefore, $\left(\left(c'_j, c_{-j}\right), \bar{s}, \bar{t}\right)$ is a continuation equilibrium, and

$$z^{\left(\left(c'_j, c_{-j}\right), \bar{s}, \bar{t}\right)} = z^{\left(\left(\hat{c}'_j, c_{-j}\right), s, t\right)},$$

i.e., c'_j is not a profitable deviation for j. Therefore, $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^F\right]}}$.

7 Other announcement and communication structures

We continue to fix $\mathcal{A} = \mathcal{A}^{non-delegated}$ but look at announcement and communication structures other than $\left\langle \Gamma^{private}, \Psi^{private} \right\rangle$. Theorem 2 in the previous section shows that the menu theorem partially holds with $\langle \Gamma, \Psi \rangle = \left\langle \Gamma^{private}, \Psi^{private} \right\rangle$ in the sense that principals can focus on menu contracts on the equilibrium path without loss of generality. However, the examples in Section 4.2 show that it is no longer the case with any other announcement and communication structures. When complex mechanisms are allowed on and off the path, there is a new equilibrium such that its equilibrium allocation cannot reproduced by any equilibrium in which principals offer menu contracts on the equilibrium path with any other announce and communication structures. Theorem 4 below shows that there is no loss of generality for principals to use menu-of-menu-with recommendation contracts on the equilibrium path and menu-of-menu-with-full-recommendation contracts off the path.

Theorem 4 *In a model $\langle \mathcal{A}, \Gamma, \Psi \rangle$ with $\mathcal{A} = \mathcal{A}^{non\text{-}delegated}$ and $\langle \Gamma, \Psi \rangle \neq \left\langle \Gamma^{private}, \Psi^{private} \right\rangle$, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^R, C^F\right]}}.$$

Theorem 4 is proved by two steps. First of all , Proposition 3 shows that there is no loss of generality to focus on menu-of-menu-with-full-recommendation contracts off the path, regardless of the complexity of contracts offered on the path and of the announcement and communication structure. Secondly, Proposition 4 shows that there is no loss of generality to focus on menu-of-menu-with-recommendation contracts as equilibrium contracts. The proofs of Propositions 3 and 4 are found in Online Appendices A and B respectively.

Proposition 3 *In the model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$ with $\mathcal{A} = \mathcal{A}^{non-delegated}$ and $\langle \Gamma,\ \Psi\rangle \neq \langle \Gamma^{private}, \Psi^{private}\rangle$, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{F}\right]}}.$$

Proposition 2 in Section 6 shows that in the model $\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\rangle$, principals focus on deviations to menu-of-menu-with-full-recommendation contracts without loss of generality to see whether a profile of menu offers can be sustained on the equilibrium path. Proposition 3 shows that the intuition behind it can be generalized in that principals focus on deviations to menu-of-menu-with-full-recommendation contracts without loss of generality to see whether a profile of any complex contracts can be sustained on the equilibrium path in the model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$ with $\mathcal{A} = \mathcal{A}^{non-delegated}$ and any other announcement and communication structure.

Proposition 4 *In the model $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle$ with $\mathcal{A} = \mathcal{A}^{non\text{-}delegated}$ and $\langle \Gamma,\ \Psi\rangle \neq \langle \Gamma^{private}, \Psi^{private}\rangle$, we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{F}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{R},C^{F}\right]}}.$$

Proposition 4 concerns the class of contracts principals can focus on the equilibrium path without loss of generality. Fix an equilibrium (c, s, t) in $\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{F}\right]}$. Suppose that the agent sends m_j to principal j given her equilibrium contract c_j. It pins down a subset of actions $c_j(m_j)$ from which principal j can choose an action (e.g., $c_j(m_j) = \{a, b, c\}$) When the announcement or communication is public, principal j's action choice from $c_j(m_j)$ generally depends on the profile of the other principals' contracts or the messages sent to the other principals. It is tempting to say that it should be enough if the contract that replaces c_j provides $\{a, b, c\}$ as its image. However, it is not. In the original equilibrium contract c_j, the agent may send different messages to principal j that lead to $\{a, b, c\}$, given a profile of the other principals' contracts and the messages that he sends to the other principals (e.g., different types of the agent may send different messages to principal j, which induce $\{a, b, c\}$). Then, principal j's action choice from $\{a, b, c\}$ may be different when the agent's different messages to principal j result in the same subset of actions, $\{a, b, c\}$. In this case, the agent's message plays a role of her non-binding action recommendation to principal j. Even if the agent sends different messages that lead to $\{a, b, c\}$, we do not need more than three messages as her non-binding action recommendations when principal j uses a pure action-choice strategy. This is the key intuition behind how we to replace principal j's equilibrium contract c_j with a menu-of-menu-with-recommendation contract $\bar{c}_j : K_j \to 2^{Y_j} \setminus \{\varnothing\}$. In particular, we specify K_j as

$$K_j = \left\{ \begin{array}{c} \left[c_j\left(m_j\right), t_j\left[\Gamma_j\left(c_{-k}, \widehat{c}'_k\right), \Psi_j\left(m_{-k}, x_k\right)\right]\right] : \\ \forall k \in \mathcal{J}, \forall \left[\widehat{c}'_k : L_k \cup H_k \to 2^{Y_j} \setminus \{\varnothing\}\right] \in C^F_k, \forall x_k \in L_k \cup H_k, \ \forall m_{-k} \in M^{\mathcal{A}}_{-k} \end{array} \right\}.$$

The proof in Online Appendix B shows how to construct an continuation equilibrium in $\mathcal{E}^{\langle \rfloor \mid \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^R, C^F]}$ for an equilibrium an equilibrium $\left(\bar{c}, \bar{s}, \bar{t}\right) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^R, C^F]}$ that reproduce the equilibrium allocation.

For the converse, fix an equilibrium (c, s, t) in $\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^R, C^F]}$. How can we construct an equilibrium contract $\bar{c}_j \in C^{\mathcal{A}}_j$ that replaces $c_j : K_j \to 2^{Y_j} \setminus \{\varnothing\}$? Because $c_j : K_j \to 2^{Y_j} \setminus \{\varnothing\}$ is in C^R_j, it satisfies the following properties: (i) $K_j \subset M^R_j$ and (ii)

$$c_j\left(\left[E_j, y_j\right]\right) = E_j, \forall \left[E_j, y_j\right] \in K_j.$$

Fix any injective function $\psi_j : K_j \longrightarrow M^{\mathcal{A}}_j$. That is, for each $\left[E_j, y_j\right] \in K_j$, each $\psi_j\left(\left[E_j, y_j\right]\right)$ fully represents $\left[E_j, y_j\right]$. We now re-label each $\left[E_j, y_j\right] \in K_j$ to $\psi_j\left(\left[E_j, y_j\right]\right) \in \psi_j\left(K_j\right)$, and replicate c_j to $\bar{c}_j|_{\psi_j(K_j)} : \psi_j\left(K_j\right) \longrightarrow \mathcal{A}_j$ such that

$$\bar{c}_j\left[\psi_j\left(\left[E_j, y_j\right]\right)\right] = E_j, \forall \left[E_j, y_j\right] \in K_j.$$

Similarly, we replicate (s, t) to $\left(\bar{s}, \bar{t}\right)$ subject to such re-labeling. Finally, we extend each $\bar{c}_j|_{\psi_j(K_j)}$ to $\bar{c}_j : M^{\mathcal{A}}_j \longrightarrow 2^{Y_j} \setminus \{\varnothing\}$ in the following way. Fix an arbitrary $\left[E^*_j, y^*_j\right] \in K_j$. Define

$$\begin{aligned} \bar{c}_j\left[\psi_j\left(\left[E_j, y_j\right]\right)\right] &= E_j, \forall \left[E_j, y_j\right] \in K_j, \\ \bar{c}_j\left[m_j\right] &= \left[E^*_j, y^*_j\right], \forall m_j \in M^{\mathcal{A}}_j \setminus \psi_j\left(K_j\right), \end{aligned}$$

i.e., $\bar{c}_j$ matches $\bar{c}_j|_{\psi_j(K_j)}$ on the sub-domain $\psi_j\left(K_j\right)$, and when the agent sends a message $m_j \in M^{\mathcal{A}}_j \setminus \psi_j\left(K_j\right)$, it is equivalent to choose $\left[E^*_j, y^*_j\right] \in K_j$. We can then extend $\left(\bar{s}, \bar{t}\right)$ subject to the extension of $\bar{c}_j|_{\psi_j(K_j)}$ to $\bar{c}_j : M^{\mathcal{A}}_j \longrightarrow 2^{Y_j} \setminus \{\varnothing\}$. Through re-labeling and extension, we can construct an equilibrium $\left(\bar{c}, \bar{s}, \bar{t}\right) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}[C^{\mathcal{A}}, C^F]}$ that reproduce the equilibrium allocation.

8 Discussion

Our results are quite general in that it allows for the agent's effort choice as the last stage of the game and can be extended for mixed strategy equilibria.

8.1 Agent's effort

Suppose that the agent chooses his effort after observing principals' actions at the last stage of the game. Let X be the set of all efforts that the agent can choose. Let $q : C^{I\cup II} \times M^{I\cup II} \times Y \times \Theta \to X$ denote the agent's effort choice strategy for an equilibrium $(c,s,t,q) \in \mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{I},C^{II}\right]}$. Therefore, $q(c,m,y,\theta)$ be the effort the agent chooses when c is the contract profile, m is the profile of messages he sends to principals, and y is the action profile that principals choose from $\left[c_j(m_j)\right]_{j\in\mathcal{J}}$.

Let $U\left(y,e,\theta\right)$ and $V_j\left(y,e,\theta\right)$ be the agent's utility and principal j's utility respectively, given $(y,e,\theta) \in Y \times X \times \Theta$. Suppose that q is Markovian in that the agent' effort choice only depends on the payoff-relevant information, i.e., for all $\theta \in \Theta$, all $y \in Y$, and all (c,m) and (c',m') such that $\left[c_j(m_j)\right]_{j\in\mathcal{J}} = \left[c'_j(m'_j)\right]_{j\in\mathcal{J}} = y$, q satisfies that

$$q(c,m,y,\theta) = q(c',m',y,\theta).$$

Then, the utility functions $u(y,\theta)$ and $v_j\left(y,\theta\right)$ specified in Section 3 are the reduced-form utility functions such that $u(y,\theta) = \max_{e\in X} U\left(y,e,\theta\right)$ and $v_j(y,\theta) = \max_{e\in X} V_j\left(y,e,\theta\right)$.

However, we believe that our results go through without restricting q to Markovian effort choice strategies. To see this concretely, consider the model with $\langle \mathcal{A},\,\Gamma,\,\Psi\rangle = \left\langle \mathcal{A}^{non-delegated},\,\Gamma^{private},\,\Psi^{private}\right\rangle$. Theorem 2 with the agent's effort choice can be established by two steps: Propositions 1 and 2. First, we can prove Proposition 1 with the agent's effort choice in the following way. To show $Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$ with the agent's effort choice, fix an equilibrium $(c,s,t,q) \in \mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}$. Because the agent's effort choice is made at the last stage after observing principals' action, we can convert (c,s,t) to $(\bar{c},\bar{s},\bar{t})$ according to the proof of $Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\,\Gamma,\,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$ in Appendix. Recall that the domain E_j of principal j's menu contract $\bar{c}_j$ is

$$E_j = \left\{ t_j\left(c_j\left(m_j\right)\right) : m_j \in M_j^{\mathcal{A}} \right\}.$$

Therefore, we can construct the agent's effort choice strategy $\overline{q}$ as

$$\overline{q}\left(\overline{c},y,y,\theta\right)=q\left(c,m,\left[t_k(c_k,m_k)\right]_{k\in\mathcal{J}},\theta\right),\forall\left(y,\theta\right)\in\left(E\times\Theta\right),$$

where $m=[m_k]_{k\in\mathcal{J}}$ is a message profile satisfying $[t_k(c_k,m_k)]_{k\in\mathcal{J}}=y$. Because of construction of E_j and the optimality of the action $q\left(c,m,\left[t_k(c_k,m_k)\right]_{k\in\mathcal{J}},\theta\right)$, the same action is also optimal for the agent after he sends y to principals, given their menu contracts $\overline{c}$. Suppose that principal j unilaterally deviates to $c'_j\in C_j^{\mathcal{A}}$. Then, for all $m_j\in M_j^{\mathcal{A}}$, all $y_j\in c_j\left(m_j\right)$, all $y_{-j}\in[E_k]_{k\neq j}$ and all $\theta\in\Theta$

$$\overline{q}\left(\left(c'_j,\overline{c}_{-j}\right),\left(m_j,y_{-j}\right),\left(y_j,y_{-j}\right),\theta\right)=q\left(\left(c'_j,c_{-j}\right),\left(m_j,m_{-j}\right),\left(y_j,\left[t_k(c_k,m_k)\right]_{k\neq j}\right),\theta\right),\tag{17}$$

where $m_{-j}=[m_k]_{k\neq j}$ is a message profile satisfying $[t_k(c_k,m_k)]_{k\neq j}=y_{-j}$. The construction of $\overline{q}$ in (17) also ensures the optimality of $\overline{q}\left(\left(c'_j,\overline{c}_{-j}\right),\left(m_j,y_{-j}\right),\left(y_j,y_{-j}\right),\theta\right)$ when the agent sends m_j to principal j, principal j chooses y_j from $c_j\left(m_j\right)$, the agent sends $y_{-j}\in[E_k]_{k\neq j}$ to the other principals. Then, $(\overline{c},\overline{s},\overline{t},\overline{q})\in Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}}$ with the same equilibrium allocation.

To show $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}}\supset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}}$ with the agent's effort choice, fix an equilibrium $(c,s,t,q)\in\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}$. Because the agent's action choice is made at the last stage after observing principals' action, we can convert (c,s,t) to $(\overline{c},\overline{s},\overline{t})$ according to the proof of $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}}\supset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}}$ in Appendix. Because $\overline{c}_j\in C_j^{\mathcal{A}}$ is constructed through re-labeling of each y_j in the domain E_j of c_j and extension of $\overline{c}_j|_{\psi_j(E_j)}$, it is also easy to construct $\overline{q}$ to show $(\overline{c},\overline{s},\overline{t},\overline{q})\in\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}$ with the same equilibrium allocation. Therefore, Proposition 1 can be established with the agent's effort choice.

Now we can establish Proposition 2 with the agent's effort choice in the following way. To show $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}}\subset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^F\right]}}$ with the agent's effort choice. First, fix equilibrium $(c,s,t,q)\in\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}$. Because the agent's action choice is made at the last stage after observing principals' action, we can convert (s,t) to $(\overline{s},\overline{t})$ according to the proof of $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^{\mathcal{A}}\right]}}\subset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^P,C^F\right]}}$ in Section 6.1.1. When no principal deviates from $c\in C^P$, the agent's effort choice strategy $\overline{q}$ is constructed according to

$$\overline{q}\left(c,y,y,\theta\right)=q\left(c,y,y,\theta\right),\forall\left(y,\theta\right)\in\left(E\times\Theta\right)$$

The optimality of $\overline{q}\left(c,y,y,\theta\right)$ is immediate. Suppose that principal j deviates to $\widehat{c}'_j:L_j\cup H_j\rightarrow 2^{Y_j}\diagdown\{\varnothing\}$ in C_j^F with $L_j\subset 2^{Y_j}$ and $H_j=\left\{\left[E_j,y_j\right]:y_j\in E_j\right\}$. Recall that we associate it with $c'_j : M_j^{\mathcal{A}} \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ for the construction of $(\bar{s}, \bar{t})$ such that given an injective function $\varphi_j : L_j \longrightarrow M_j^{\mathcal{A}}$,

$$c'_j \left[\varphi_j \left(D_j \right) \right] = D_j, \forall D_j \in L_j,$$
$$c'_j \left(m_j \right) = E_j, \forall m_j \in M_j^{\mathcal{A}} \diagdown \varphi_j \left(L_j \right).$$

For all $x_j \in L_j \cup H_j$, let

$$\xi(x_j) = \begin{cases} \varphi_j \left(x_j \right) & \text{if } x_j \in L_j, \\ \overline{m}_j \in M_j^{\mathcal{A}} \diagdown \varphi_j \left(L_j \right) & \text{otherwise.} \end{cases}$$

Finally, we construct $\bar{q}$ upon principal j's deviation to $\widehat{c}'_j$ as follows: For all $x_j \in L_j \cup H_j$, all possible y_j given x_j, all possible y_{-j} given $[c_k]_{k \neq j}$, all possible $\theta \in \Theta$,

$$\bar{q} \left(\left(\widehat{c}'_j, c_{-j} \right), \left(x_j, y_{-j} \right), \left(y_j, y_{-j} \right), \theta \right) = q \left(\left(c'_j, c_{-j} \right), \left(\xi(x_j), y_{-j} \right), \left(y_j, y_{-j} \right), \theta \right)$$

The optimality of $\bar{q} \left(\left(\widehat{c}'_j, c_{-j} \right), \left(x_j, y_{-j} \right), \left(y_j, y_{-j} \right), \theta \right)$ is ensured because of the optimality $q \left(\left(c'_j, c_{-j} \right), \left(\xi(x_j), y_{-j} \right), \left(y_j, y_{-j} \right), \theta \right)$. Then, $(c, \bar{s}, \bar{t}, \bar{q}) \in Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^F \right]}}$ with the same equilibrium allocation.

To show $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^{\mathcal{A}} \right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^F \right]}}$ with the agent's effort choice, fix an equilibrium $(c, s, t, q) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^F \right]}$. Because the agent's action choice is made at the last stage after observing principals' action, we can convert (s, t) to $(\bar{s}, \bar{t})$ according to the proof of $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^{\mathcal{A}} \right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^P, C^F \right]}}$ in Section 6.1.2. Recall that for any deviation of $c'_j \in C_j^{\mathcal{A}}$, we associate it with $\widehat{c}'_j \in C_j^F$ such that c'_j is induced by the re-labeling and extension, it is also easy to construct $\bar{q}$ to show $(c, \bar{s}, \bar{t}, \bar{q}) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}} \right]}$ with the same equilibrium allocation. Therefore, Proposition 2 can be established with the agent's effort choice and subsequently, Theorem 2 can be established with the agent's effort choice. We believe that Theorem 4 for any other announcement and communication structures can be established with the agent's effort choice as well.

8.2 Mixed-strategy equilibria

We adopt the notion of pure-strategy equilibria in which all players employ pure strategies. Our results can be extended for mixed-strategy equilibria. To see this concretely, let us take the model with $\langle \mathcal{A},\, \Gamma,\, \Psi \rangle = \langle \mathcal{A}^{non-delegated},\, \Gamma^{private},\, \Psi^{private} \rangle$. Principals can still use menu contracts on the equilibrium path without loss of generality. Off the path, they can also still use menu-of-menu-with-full-recommendation contracts but we should extend them to incorporate a principal's randomization over actions from a subset of actions that is determined by the agent's message. To see this, consider principal j's unilateral deviation to a contract $c_j : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}$ such that

$$\begin{aligned} c_j'(m_j') &= \{a,b\}\,, c_j'(m_j'') = \{c,d,e\}\,, \\ c_j'(m_j) &= \{f,g\}\,, \forall m_j \in M_j^{\mathcal{A}} \diagdown \{m_j', m_j''\}. \end{aligned}$$

Given the agent's message m_j that leads to $c_j'(m_j) = \{f,g\}$, suppose that principal j chooses f with probability p and g with probability $1-p$. If the agent also randomizes her message that leads to $c_j'(m_j) = \{f,g\}$, we cannot convert c_j' to a menu-of-menu-with-full-recommendation contract $\widehat{c}_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ with

$$L_j \cup H_j = \{\{a,b\}\,, \{c,d,e\}\,, [\{f,g\}\,, f]\,, [\{f,g\}\,, g]\}$$

Why? Let $t_j(c_j', m_j)$ be the probability that principal j chooses f when receiving $m_j \in M_j^{\mathcal{A}} \diagdown \{m_j', m_j''\}$. Suppose that $\{m_j', m_j''\} \cup M_j$ is the support of the probability distribution that the agent uses for randomizing her message to principal j, where M_j is a non-singleton subset of $M_j^{\mathcal{A}} \diagdown \{m_j', m_j''\}$. Then, we should convert c_j' to an extended menu-of-menu-with-full-recommendation contract $\widehat{c}_j : L_j \cup H_j' \to 2^{Y_j} \diagdown \{\varnothing\}$ with $L_j = \{\{a,b\}\,, \{c,d,e\}\}$ and

$$H_j' = \{[\{f,g\}\,, p]\,; p \in R_j\}\,,$$

where $R_j \subset [0,1]$ has its cardinality equal to

$$\left| \left\{ t_j(c_j', m_j) : m_j \in M_j \right\} \right|.$$

That is, the possible recommendations associated with $\{f,g\}$ must be as many as the possible values of for the probability that principal j chooses f.

When the announcement or communication (or both) are public, the menu-of-menu-with-recommendation contracts on the equilibrium path should be extended analogously. These extensions show the inherent intractability of mixed-strategy equilibrium analysis in common agency without delegation: The message space in the extended menu-of-menu-with-(full or not)-recommendation contract can be countably infinite even though the action space is countably finite because there is no a priori reason to believe that R_j is countably finite. Such an intractability does not arise with pure-strategy equilibria.

9 Conclusion

In this paper, we study common agency without delegation in order to address Szentes' critique (Szentes (2009)). We show that the menu theorem in Peters (2001) holds only partially under private announcement and private communication, and that it fails generally. Finally, we prove a menu-of-menu-with-recommendation theorem for common agency without delegation.

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Appendix. Proof of Proposition 1

Proof of $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$

We first show $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$. Fix any $(c,s,t) \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}$, i.e., $c \in C^{P}$, and (c,s,t) is an equilibrium subject to any principal j's deviation to contracts in $C_j^{\mathcal{A}}$. We aim to construct an equilibrium $\left[\bar{c},\bar{s},\bar{t}\right] \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}$ such that $z^{\left[\bar{c},\bar{s},\bar{t}\right]} = z^{(c,s,t)}$.

We show this by two steps. For the first step, since c_j is in C_j^P for each $j \in \mathcal{J}$, we have a pair of (i) $E_j \subset 2^{Y_j} \diagdown \{\varnothing\}$ and (ii) $c_j : E_j \longrightarrow Y_j$ such that

$$c_j\left(y_j\right) = y_j, \forall y_j \in E_j.$$

Fix any injective function $\psi_j : E_j \longrightarrow M_j^{\mathcal{A}}$. That is, for each $y_j \in E_j$, each $\psi_j\left(y_j\right)$ fully represents y_j. We now re-label each $y_j \in E_j$ to $\psi_j\left(y_j\right) \in \psi_j\left(E_j\right)$, and replicate c_j to $\bar{c}_j|_{\psi_j(E_j)} : \psi_j\left(E_j\right) \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ such that

$$\bar{c}_j\left[\psi_j\left(y_j\right)\right] = y_j, \forall y_j \in E_j.$$

Similarly, we replicate (s,t) to $\left(\bar{s},\bar{t}\right)$ subject to such re-labeling. Specifically, $\bar{s}$ is defined as follows.

$$\bar{s}_j\left(\bar{c},\theta\right) = \psi_j\left[s_j\left(c,\theta\right)\right], \forall j \in \mathcal{J}, \forall \theta \in \Theta. \tag{18}$$

i.e., when $\bar{c}$ is a profile of contracts in $C^{\mathcal{A}}$ that principals offer, the agent of type θ would send the message $\psi_j\left[s_j\left(c,\theta\right)\right]$ to principal j, which results in the action $s_j\left(c,\theta\right) \in E_j$ that the agent would choose under c_j. If j deviates to $c_j' \in C_j^{\mathcal{A}}$, we have

$$\begin{aligned} \bar{s}_j\left(\left(c_j',\bar{c}_{-j}\right),\theta\right) &= s_j\left(\left(c_j',c_{-j}\right),\theta\right), \forall \theta \in \Theta, &(19)\\ \bar{s}_k\left(\left(c_j',\bar{c}_{-j}\right),\theta\right) &= \psi_k\left[s_k\left(\left(c_j',c_{-j}\right),\theta\right)\right], \forall k \neq j, \forall \theta \in \Theta, &(20)\end{aligned}$$

that is, $\bar{s}$ follows s.

Rigorously, for each $j \in \mathcal{J}$, principal j's action choice strategy $\bar{t}_j$ and her belief $\bar{b}_j$ are defined as follows. If every j does not deviate from $\bar{c}_j$, we have

$$\begin{aligned} \bar{t}_j\left(\bar{c}_j,\psi_j\left(y_j\right)\right) &= t_j\left(c_j,y_j\right) = y_j, \forall y_j \in E_j. &(21)\\ \bar{b}_j\left(\bar{c}_j,\psi_j\left(y_j\right)\right) &= b_j\left(c_j,y_j\right), \forall y_j \in E_j\end{aligned}$$

i.e., j chooses the action $\bar{c}_j\left[\psi_j\left(y_j\right)\right] = y_j$ she would choose when the agent directly chooses y_j under c_j. The belief $\bar{b}_j\left(\bar{c}_j, \psi_j\left(y_j\right)\right)$ clearly satisfies the requirement regarding Bayes' rule (i.e., Bayes' rule applies only when principal j cannot confirm that a player has deviated) because $b_j\left(c_j, y_j\right)$ satisfies it, but the belief is not relevant for principal j's action choice is not strategic.

If j unilaterally deviates to $\left[c'_j : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}\right] \in C_j^{\mathcal{A}}$, we have that for all $j \in \mathcal{J}$

$$\bar{t}_j\left(c'_j, m_j\right) = t_j\left(c'_j, x_j\right), \forall m_j \in M_j^{\mathcal{A}}, \tag{22a}$$

$$\bar{b}_j\left(c'_j, x_j\right) = b_j\left(c'_j, x_j\right), \forall m_j \in M_j^{\mathcal{A}}, \tag{22b}$$

i.e., $\bar{t}_j$ replicates t_j, and $\bar{b}_j$ also replicates b_j for all $j \in \mathcal{J}$. The belief $\bar{b}_j\left(c'_j, x_j\right)$ satisfies Bayes' rule because $b_j\left(c'_j, x_j\right)$ satisfies it.

That is, $\left[\left(\bar{c}_j|_{\psi_j(E_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ replicates (c, s, t) subject to re-labeling each $y_j \in E_j$ to $\psi_j\left(y_j\right) \in \psi_j\left(E_j\right)$. Clearly, the former inherits incentive compatibility of the latter, and hence, $\left[\left(\bar{c}_j|_{\psi_j(E_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ is an equilibrium subject any principal j's deviation to contracts in $C_j^{\mathcal{A}}$. Subsequently, we have

$$z^{\left[\left(\bar{c}_j|_{\psi_j(E_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \left(\bar{t}_j\right)_{j\in\mathcal{J}}\right]} = z^{(c,s,t)}. \tag{23}$$

In Step 2, we extend each $\bar{c}_j|_{\psi_j(E_j)}$ to $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$. Fix an arbitrary $y'_j \in E_j$. Define

$$\bar{c}_j\left[\psi_j\left(y_j\right)\right] = y_j, \forall y_j \in E_j,$$

$$\bar{c}_j\left[m_j\right] = y'_j, \forall m_j \in M_j^{\mathcal{A}} \diagdown \psi_j\left(E_j\right),$$

i.e., $\bar{c}_j$ matches $\bar{c}_j|_{\psi_j(E_j)}$ on the sub-domain $\psi_j\left(E_j\right)$, and when the agent sends a message $m_j \in M_j^{\mathcal{A}} \diagdown \psi_j\left(E_j\right)$, it is equivalent to choosing $y'_j \in E_j$. We also need to extend $\bar{t}_j$ and $\bar{b}_j$ when principal chooses $\bar{c}_j$ (rather than $\bar{c}_j|_{\psi_j(E_j)}$) as follows. For all $j \in \mathcal{J}$ and all $m_j \in M_j^{\mathcal{A}}$, define $\psi_j^{-1}\left(m_j\right)$ as

$$\psi_j^{-1}\left(m_j\right) = \begin{cases} y_j & \text{if } \exists y_j \text{ s.t. } m_j = \psi_j(y_j) \\ y'_j & \text{otherwise, i.e.,} m_j \in M_j^{\mathcal{A}} \backslash \psi_j(E_j) \end{cases}$$

Given the extension $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ for all $j \in \mathcal{J}$, principal j's action choice and her belief follow

$$\bar{t}_j\left(\bar{c}_j, m_j\right) = t_j\left(c_j, \psi_j^{-1}\left(m_j\right)\right), \forall m_j \in M_j^{\mathcal{A}},$$
$$\bar{b}_j\left(\bar{c}_j, m_j\right) = b_j\left(c_j, \psi_j^{-1}\left(m_j\right)\right), \forall m_j \in M_j^{\mathcal{A}}.$$

It is clear that the belief $\bar{b}_j\left(\bar{c}_j, m_j\right)$ satisfies the requirement of Bayes' rule. Upon principal j's deviation to $\left[c_j' : M_j^{\mathcal{A}} \to 2^{Y_j} \diagdown \{\varnothing\}\right] \in C_j^{\mathcal{A}}$, principal j's action choice and her belief follow (22a) and (22b).

The extensions of $\bar{c}_j|_{\psi_j(E_j)}$ to $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ for all $j \in \mathcal{J}$ do not change the incentive compatibility of $\bar{s}$ for the agent: under $\bar{c}_j|_{\psi_j(E_k)}$, choosing $\psi_j\left(y_j'\right) = y_j'$ is not a profitable deviation for the agent, and hence, under $\bar{c}_j$, choosing $m_j \in M_j^{\mathcal{A}} \diagdown \psi_j\left(E_j\right)$ will not be a profitable deviation for the agent either. Furthermore, given the agent's communication strategy $\bar{s}$, the extension of $\bar{t}_j$ is also the best reply for principal j given the extension of $\bar{b}_j$ for all $j \in \mathcal{J}$. Therefore, $\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ is a $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-equilibrium that replicates $\left[\left(\bar{c}_j|_{\psi_j(E_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$:

$$z^{\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \left(\bar{t}_j\right)_{j\in\mathcal{J}}\right]} = z^{\left[\left(\bar{c}_j|_{\psi_j(E_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \left(\bar{t}_j\right)_{j\in\mathcal{J}}\right]}, \tag{24}$$

$$\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \left(\bar{t}_j\right)_{j\in\mathcal{J}}\right] \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}. \tag{25}$$

(23) and (24) imply

$$z^{\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \left(\bar{t}_j\right)_{j\in\mathcal{J}}\right]} = z^{(c,s,t)}. \tag{26}$$

Therefore, (25) and (26) imply $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$.

Proof of $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$

We now show $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}}$. Fix any $(c,s,t) \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^{\mathcal{A}}\right]}$, and we aim to construct an equilibrium $\left[\bar{c}, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{P},C^{\mathcal{A}}\right]}$ such that $z^{\left[\bar{c},\bar{s},\bar{t}\right]} = z^{(c,s,t)}$.

We first define $\bar{c}_j$ for each principal j. Define

$$E_j \equiv \left\{t_j\left(c_j\left(m_j\right)\right) : m_j \in M_j^{\mathcal{A}}\right\},$$

i.e., under the equilibrium (c, s, t), for each $m_j \in M_j^{\mathcal{A}}$ sent from the agent to j, it pins down $c_j(m_j) \in 2^{Y_j} \diagdown \{\varnothing\}$ and j chooses $t_j(c_j(m_j))$. Therefore, offering c_j under the equilibrium (c, s, t) is equivalent to offering a menu of E_j, and the agent chooses the best y_j in E_j. This is the insight of Peters (2001) and Martimort and Stole (2002). Hence, we define $\bar{c}_j : E_j \longrightarrow Y_j$ as

$$\bar{c}_j(y_j) = y_j, \forall y_j \in E_j.$$

We construct $\bar{s}$ that replicate s subject to $\bar{c}$ replicating c. Fix any $j \in \mathcal{J}$. If no principal deviates, we have

$$\bar{s}_j(\bar{c}, \theta) = t_j\left[c_j, s_j(c, \theta)\right], \forall j \in \mathcal{J}, \forall \theta \in \Theta,$$

i.e., the agent of type θ would recommend $t_j\left[s_j(c, \theta)\right]$ to principal j under $\bar{c}_j$, which is the action she would choose under c_j. If j deviates to $c'_j \in C_j^{\mathcal{A}}$, we have

$$\begin{aligned}
\bar{s}_j\left(\left(c'_j, \bar{c}_{-j}\right), \theta\right) &= s_j\left(\left(c'_j, c_{-j}\right), \theta\right), \forall \theta \in \Theta, \\
\bar{s}_k\left(\left(c'_j, \bar{c}_{-j}\right), \theta\right) &= t_k\left[c_k, s_j\left(\left(c'_j, c_{-j}\right), \theta\right)\right], \forall k \neq j, \forall \theta \in \Theta,
\end{aligned}$$

i.e., $\bar{s}_j$ follows s_j and what the agent chooses from non-deviating principal k's menu $\bar{c}_k$ is the same as principal k's action choice under c_k when she receives $s_j\left(\left(c'_j, c_{-j}\right), \theta\right)$.

Similarly, $\bar{t}$ replicates t (and $\bar{b}$ replicates b). Rigorously, for each $j \in \mathcal{J}$. If principal j does not deviate from $\bar{c}_j$, principal j's action choice follows

$$\bar{t}_j(\bar{c}_j, y_j) = y_j, \forall y_j \in E_j,$$

i.e., j follows the agent's recommendation. Since $\bar{c}_j$ is a menu contract, the belief is irrelevant but for the completeness of the proof, we construct it. Given $\bar{c}_j$, suppose that principal j receives a message y_j with $\{\theta'' \in \Theta : y_j = t_j(s_j((c_j, c_{-j}), \theta''))\} \neq \varnothing$. Then principal j's belief satisfies that for all $\theta \in \left\{\theta'' \in \Theta : t_j\left(s_j\left(\left(c_j, c_{-j}\right), \theta'\right)\right) = y_j\right\}$

$$\bar{b}_j(\bar{c}_j, y_j)(\bar{c}_j, \bar{c}_{-j}, \theta) = \frac{p(\theta)}{\sum_{\theta' \in \left\{\theta'' \in \Theta : t_j\left(s_j\left(\left(c_j, c_{-j}\right), \theta'\right)\right) = y_j\right\}} p(\theta'')}.$$

If principal j receives a message y_j with $\left\{\theta'' \in \Theta : t_j\left(s_j\left(\left(c_j, c_{-j}\right), \theta''\right)\right) = y_j\right\} = \varnothing$, then choose an arbitrary belief $\bar{b}_j(\bar{c}_j, y_j)$.

If j deviates to $c'_j \in C_j^{\mathcal{A}}$, we have

$$\begin{aligned}
\bar{t}_j\left(c'_j, m_j\right) &= t_j\left(c'_j, m_j\right), \forall x_j \in M_j^{\mathcal{A}}, \\
\bar{b}_j\left(c'_j, m_j\right) &= b_j\left(c'_j, m_j\right), \forall x_j \in M_j^{\mathcal{A}}
\end{aligned}$$

i.e., $\bar{t}_j\left(c'_j, x_j\right)$ replicates $\bar{t}_j\left(c'_j, x_j\right)$ given the same belief. The belief $b_j\left(c'_j, x_j\right)$ satisfies the requirement of Bayes' rule because $b_j\left(c'_j, x_j\right)$ satisfies it.

As shown above, $\left[\bar{c}, \bar{s}, \bar{t}\right]$ replicates the allocation of (c, s, t) regarding any possible unilateral deviations by principals i.e.,

$$z^{\left[\left(c'_j, \bar{c}_{-j}\right), \bar{s}, \bar{t}\right]} = z^{\left(\left(c'_j, c_{-j}\right), s, t\right)}, \forall j, \forall c'_j \in C_j^{\mathcal{A}}.$$

Hence, $\left[\bar{c}, \bar{s}, \bar{t}\right]$ inherits the incentive compatibility of principals and the agent from that of (c, s, t) on and off the path following a player's unilateral deviation. As a result, $z^{\left[\bar{c}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$ and $\left[\bar{c}, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}$. Therefore, $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^P, C^{\mathcal{A}}\right]}}$.

Online Appendix A. Proof of Proposition 3

Proof of $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{F}\right]}}$

Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}}, C^{\mathcal{A}}]}$, and we aim to construct an equilibrium $\left(c, \bar{s}, \bar{t}\right) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}}, C^{F}]}$ such that $z^{\left(c, \bar{s}, \bar{t}\right)} = z^{(c,s,t)}$.

First, define

$$\bar{s}\left(c, \theta\right) = s\left(c, \theta\right), \forall \theta \in \Theta,$$

$$\begin{aligned} \bar{t}_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right) &= t_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right), \forall j \in \mathcal{J}, \forall m \in M^{\mathcal{A}}, \\ \bar{b}_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right) &= b_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right), \forall j \in \mathcal{J}, \forall m \in M^{\mathcal{A}}. \end{aligned}$$

i.e., if no principal deviates from c, we let $\left(\bar{s}, \bar{t}\right)$ be equal to (s, t). As a result, $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$, the agent's and principals' incentive compatibility is preserved on the equilibrium path (i.e., when no principal deviates from c) given the same belief, which clearly satisfies Bayes' rule.

Second, for any deviation of $\widehat{c}_j' \in C_j^F$ by principal j, we must construct a continuation equilibrium such that this deviation is not profitable for j. Since $\widehat{c}_j' \in C_j^F$, we have a pair of (i) $\left[L_j \subset 2^{Y_j} \text{ and } H_j = \left\{\left[E_j, y_j\right] : y_j \in E\right\}\right]$ and (ii) a mapping, $\widehat{c}_j' : L_j \cup H_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ such that $\phi\left(H_j\right) \notin L_j$ and

$$\begin{aligned} \widehat{c}_j'\left(D_j\right) &= D_j, \forall D_j \in L_j, \\ \widehat{c}_j'\left(\left[E_j, y_j\right]\right) &= E_j, \forall \left[E_j, y_j\right] \in H_j. \end{aligned}$$

Fix any injective function $\varphi_j : L_j \longrightarrow M_j^{\mathcal{A}}$. That is, for each $D_j \in L_j$, the message $\varphi_j\left(D_j\right) \in M_j^{\mathcal{A}}$ fully represents D_j. We now define a contract $c_j' : M_j^{\mathcal{A}} \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ as follows.

$$c_j'\left[\varphi_j\left(D_j\right)\right] = D_j, \forall D_j \in L_j,$$

$$c_j'\left(m_j\right) = E_j, \forall m_j \in M_j^{\mathcal{A}} \diagdown \varphi_j\left(K_j\right).$$

That is, for every $D_j \in L_j$, the message $\varphi_j\left(D_j\right)$ pins down the subset $D_j \subset Y_j$, and any other messages (i.e., $m_j \in M_j^{\mathcal{A}} \diagdown \varphi_j\left(L_j\right)$) pins down the subset $E_j \subset Y_j$.

Clearly, $c'_j \in C_j^{\mathcal{A}}$. Since $(c,s,t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}[C^{\mathcal{A}},C^{\mathcal{A}}]}$, c'_j is not a profitable deviation under (c,s,t). Based on this, we construct $[\bar{s},\bar{t}]$ as follows.

$$\begin{pmatrix} s_j\left(\left(c'_j,c_{-j}\right),\theta\right)=\varphi_j\left(D_j\right), \\ D_j\in L_j \end{pmatrix} \Longrightarrow \bar{s}_j\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)=D_j, \forall\theta\in\Theta, \tag{27}$$

$$\begin{pmatrix} s_j\left(\left(c'_j,c_{-j}\right),\theta\right)\in M_j^{\mathcal{A}}\diagdown\varphi_j\left(L_j\right), \\ t_j\left[\Gamma_j\left(c'_j,c_{-j}\right),\Psi_j\left(s\left(\left(c'_j,c_{-j}\right),\theta\right)\right)\right]=y_j\in E_j \end{pmatrix} \Longrightarrow$$
$$\bar{s}_j\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)=\left[E_j,y_j\right], \forall\theta\in\Theta, \tag{28}$$

$$\bar{s}_{-j}\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)=s_{-j}\left(\left(c'_j,c_{-j}\right),\theta\right), \forall\theta\in\Theta \tag{29}$$

(27) shows that if the agent's message pins down $D_j\in L_j$ under $\left(c'_j,c_{-j}\right)$, we let the agent send the corresponding message that pins $D_j\in L_j$ under $\left(\widehat{c}'_j,c_{-j}\right)$. (28) shows that if the agent's message pins down E_j under $\left(c'_j,c_{-j}\right)$ and principal j subsequently chooses $y_j\in E_j$ under t_j, we let the agent send the corresponding message that pins $\left[E_j,y_j\right]\in H_j$ under $\left(\widehat{c}'_j,c_{-j}\right)$.

For all $k\in\mathcal{J}$, $\bar{t}_k$ and $\bar{b}_k$ replicate t_k and b_k respectively upon principal j's deviation to $\widehat{c}'_j\in C_j^F$. First of all,

$$\bar{t}_k\left(\Gamma_k\left(\widehat{c}'_j,c_{-j}\right),\Psi_k\left(D_j,m_{-j}\right)\right)= \tag{30}$$
$$t_k\left[\Gamma_k\left(c'_j,c_{-j}\right),\Psi_k\left(\varphi_j\left(D_j\right),m_{-j}\right)\right], \forall k\in\mathcal{J}, \forall D_j\in L_j, \forall m_{-j}\in M_{-j}^{\mathcal{A}},$$
$$\bar{b}_k\left(\Gamma_k\left(\widehat{c}'_j,c_{-j}\right),\Psi_k\left(D_j,m_{-j}\right)\right)= \tag{31}$$
$$b_k\left[\Gamma_k\left(c'_j,c_{-j}\right),\Psi_k\left(\varphi_j\left(D_j\right),m_{-j}\right)\right], \forall k\in\mathcal{J}, \forall D_j\in L_j, \forall m_{-j}\in M_{-j}^{\mathcal{A}}.$$

i.e., when the agent's message pins down $D_j\in L_j$, principal k chooses the same action under $\bar{t}_k$ given the same belief $\bar{b}_k$ as b_k. Note that the belief $\bar{b}_k$ in (31) satisfies the requirement of Bayes' rule as b_k satisfies it.

Secondly, by (28), the set of messages under $\bar{s}_j$ that could be sent from some of the agent's types to principal j is defined as follows.

$$Q_j\equiv\left\{\left[E_j,y_j\right]: \begin{matrix} s_j\left(\left(c'_j,c_{-j}\right),\theta\right)\in M_j^{\mathcal{A}}\diagdown\varphi_j\left(L_j\right), \\ t_j\left[\Gamma_j\left(c'_j,c_{-j}\right),\Psi_j\left(s\left(\left(c'_j,c_{-j}\right),\theta\right)\right)\right]=y_j\in E_j \end{matrix}\right\}.$$

Note that every element in Q_j has E_j in it. We define $\bar{t}_k$ and $\bar{b}_k$ for all $k \in \mathcal{J}$ when the agent sends the message profile $([E_j, y_j], m_{-j}) \in Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$ upon j's deviation to $\widehat{c}_j'$ as follows, that is,

$$\bar{t}_k\left(\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left([E_j, y_j], s_{-j}\left(\left(c_j', c_{-j}\right), \theta\right)\right)\right) = \\ t_k\left[\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(s\left(\left(c_j', c_{-j}\right), \theta\right)\right)\right], \forall [E_j, y_j] \in Q_j, \quad (32)$$

given the belief

$$\bar{b}_k\left(\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left([E_j, y_j], s_{-j}\left(\left(c_j', c_{-j}\right), \theta\right)\right)\right) = \\ b_k\left[\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(s\left(\left(c_j', c_{-j}\right), \theta\right)\right)\right], \forall [E_j, y_j] \in Q_j. \quad (33)$$

If $k = j$, then $\bar{b}_k$ in (33) satisfies the requirement of Bayes' rule since b_k satisfies it. (28) says that, if the agent's message pins down E_j under $\left(c_j', c_{-j}\right)$ and principal j subsequently chooses $y_j \in E_j$ under t_j, we let the agent send the corresponding message that pins $[E_j, y_j] \in H_j$ under $\left(\widehat{c}_j', c_{-j}\right)$. (32) says that principal j indeed follows the agent's recommendation of y_j given the same belief as b_j because $[E_j, y_j]$ is in Q_j and y_j is the action she would choose under b_j, i.e., $t_j\left[\Gamma_j\left(c_j', c_{-j}\right), \Psi_j\left(s\left(\left(c_j', c_{-j}\right), \theta\right)\right)\right] = y_j \in E_j$. Thus, $z^{\left[\left(\widehat{c}_j', c_{-j}\right), \bar{s}, \bar{t}\right]}$ indeed replicates the $z^{\left(\left(c_j', c_{-j}\right), s, t\right)}$, and as a result, $\left[\left(\widehat{c}_j', c_{-j}\right), \bar{s}, \bar{t}\right]$ inherits the agent's and principal j's incentive compatibility of $z^{\left(\left(c_j', c_{-j}\right), s, t\right)}$.

Finally, we define $\bar{t}_k$ and $\bar{b}_k$ for all $k \in \mathcal{J}$ when the agent sends the message profile $([E_j, y_j], m_{-j})$, which is in $H_j \times M_{-j}^{\mathcal{A}}$ but not in $Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$ upon j's deviation to $\widehat{c}_j'$. For all $[E_j, y_j] \in H_j$, let $m_j^* \in M_j^{\mathcal{A}}$ be an arbitrary message such that $c_j\left(m_j^*\right) = E_j$.

$$\forall ([E_j, y_j], m_{-j}) \in H_j \times M_{-j}^{\mathcal{A}}, \text{ if } ([E_j, y_j], m_{-j}) \notin Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right), \\ \bar{t}_k\left(\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left([E_j, y_j], m_{-j}\right)\right) = t_k\left[\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j^*, m_{-j}\right)\right], \quad (34)$$

given the belief

$$\forall ([E_j, y_j], m_{-j}) \in H_j \times M_{-j}^{\mathcal{A}}, \text{ if } ([E_j, y_j], m_{-j}) \notin Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right), \\ \bar{b}_k\left(\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left([E_j, y_j], m_{-j}\right)\right) = b_k\left[\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j^*, m_{-j}\right)\right], \quad (35)$$

Suppose that the agent sends a profile of messages $([E_j, y_j], m_{-j})$ in $H_j \times M_{-j}^{\mathcal{A}}$ but not in $Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$. We pick an arbitrary message $m_j^* \in M_j^{\mathcal{A}}$ such that $c_j\left(m_j^*\right) =$

E_j. Then, we replicate t_k and b_k when the message profile is $\left(m_j^*, m_{-j}\right)$ with $\bar{t}_k$ and $\bar{b}_k$ with such a profile of messages $\left(\left[E_j, y_j\right], m_{-j}\right)$, which is in $H_j \times M_{-j}^{\mathcal{A}}$ but not in $Q_j \times s_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$. For the belief, note that whenever principal k ($\neq j$) cannot confirm the other players' unilateral deviations, she cannot confirm the other players' unilateral deviations. Further, whenever principal j cannot confirm the agent's deviation given her contract $\widehat{c}_j'$, she cannot also confirm the agent's deviation in the original continuation equilibrium given her contract c_j' (Since we consider only a unilateral deviation by one principal, the other principals are assumed to continue to offer their equilibrium contracts). Therefore, (35) satisfies the requirement of the belief: Bayes' rule is applied only when another player's deviation cannot be confirmed.

Because $\left(\left(c_j', c_{-j}\right), s, t\right)$ specifies a continuation equilibrium upon j's deviation to c_j' given c_{-j}, our construction of strategy profiles and beliefs shows (i) $\left(\left(\widehat{c}_j', c_{-j}\right), \bar{s}, \bar{t}\right)$ also specifies a continuation equilibrium upon j's deviation to $\widehat{c}_j'$ given c_{-j} and (ii) $z^{\left[\left(\widehat{c}_j', c_{-j}\right), \bar{s}, \bar{t}\right]} = z^{\left(\left(c_j', c_{-j}\right), s, t\right)}$. Therefore, deviation to $\widehat{c}_j'$ is not profitable. Thus, $\left[c, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}$ and $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$. ■

Proof $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}}$

Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}$, and we aim to construct an equilibrium $\left[c, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$ such that $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$.

First, define

$$\bar{s}\left(c, \theta\right) = s\left(c, \theta\right), \forall \theta \in \Theta,$$

$$\begin{aligned} \bar{t}_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right) &= t_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right), \forall j \in \mathcal{J}, \forall m \in M^{\mathcal{A}}, \\ \bar{b}_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right) &= b_j\left(\Gamma_j\left(c\right), \Psi_j(m)\right), \forall j \in \mathcal{J}, \forall m \in M^{\mathcal{A}}. \end{aligned}$$

i.e., if no principal deviates from c, we let $\left[\bar{s}, \bar{t}\right]$ be equal to $[s, t]$. As a result, $z^{\left[c, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$, the agent's and principals' incentive compatibility is preserved when no principal deviates from c.

Second, for any deviation of $c_j' \in C_j^{\mathcal{A}}$ by principal j, we must construct a continuation equilibrium such that this deviation is not profitable for j. Consider

$$\mathcal{L}_j \equiv \left\{ c'_j\left(m_j\right) \in 2^{Y_j} \diagdown \{\varnothing\} : m_j \in M_j^{\mathcal{A}} \right\}.$$

Since $M_j^{\mathcal{A}}$ has a much larger cardinality than $2^{Y_j} \diagdown \{\varnothing\}$, there exists $E_j^* \in \mathcal{L}_j$ such that

$$\left| \left\{ m_j \in M_j^{\mathcal{A}} : c'_j\left(m_j\right) = E_j^* \right\} \right| > \left| E_j^* \right|.$$

Consider $\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ defined as follows.

$$\begin{aligned} L_j &= \mathcal{L}_j \diagdown \left\{ E_j^* \right\}, \\ H_j &= \left\{ \left[E_j^*, y_j \right] : y_j \in E_j^* \right\}. \end{aligned}$$

and

$$\begin{aligned} \widehat{c}'_j\left(D_j\right) &= D_j, \forall D_j \in L_j, \\ \widehat{c}'_j\left(\left[E_j^*, y_j \right]\right) &= E_j^*, \forall \left[E_j^*, y_j \right] \in H_j. \end{aligned}$$

Clearly, $\widehat{c}'_j \in C_j^F$. Since $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}$, $\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)$ is a continuation equilibrium, and it is not profitable for j.

Fix any injective function $\varphi_j : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}$, such that

$$\begin{aligned} c'_j\left[\varphi_j\left(D_j\right)\right] &= D_j, \forall D_j \in L_j, \\ c'_j\left[\varphi_j\left(\left[E_j^*, y_j \right]\right)\right] &= E_j^*, \forall \left[E_j^*, y_j \right] \in H_j. \end{aligned}$$

That is, for each $D_j \in L_j$, we pick a message $\varphi_j\left(D_j\right) \in M_j^{\mathcal{A}}$ such that $\varphi_j\left(D_j\right)$ pins down D_j under c'_j. Furthermore, for each $\left[E_j^*, y_j \right] \in H_j$, we pick a distinct message $\varphi_j\left(\left[E_j^*, y_j \right]\right) \in M_j^{\mathcal{A}}$ such that $\varphi_j\left(\left[E_j^*, y_j \right]\right)$ pins down E_j^* under c'_j.

Hypothetically, let us delete messages in $M_j^{\mathcal{A}} \diagdown \varphi_j\left(L_j \cup H_j\right)$, i.e., we consider the contract c'_j with the restricted domain of $\varphi_j\left(L_j \cup H_j\right)$, which is denoted by $c'_j|_{\varphi_j\left(L_j \cup H_j\right)}$. As a result, the contract $\widehat{c}'_j$ is equivalent to the contract $c'_j|_{\varphi_j\left(L_j \cup H_j\right)}$ subject to re-labeling of messages via the bijection φ_j, which maps elements in $L_j \cup H_j$ to elements in $\varphi_j\left(L_j \cup H_j\right)$. Thus, we can define $\left(\left(c'_j|_{\varphi_j\left(L_j \cup H_j\right)}, c_{-j}\right), \overline{s}, \overline{t}\right)$ to replicate $\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)$, and therefore $\left(\left(c'_j|_{\varphi_j\left(L_j \cup H_j\right)}, c_{-j}\right), \overline{s}, \overline{t}\right)$ is a continuation equilibrium, which inherits incentive compatibility of the agent and the principals under $\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)$. Rigorously, we define $\left(\overline{s}, \overline{t}\right)$

as follows.

$$\begin{aligned} s_j\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right) &= x_j \Longrightarrow \bar{s}_j\left(\left(c_j', c_{-j}\right), \theta\right) = \varphi_j\left(x_j\right), \forall x_j \in L_j \cup H_j, \forall \theta \in \Theta, \\ \bar{s}_{-j}\left(\left(c_j', c_{-j}\right), \theta\right) &= s_{-j}\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right), \forall \theta \in \Theta \end{aligned}$$

$$\begin{aligned} &\bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j(x_j), m_{-j}\right)\right) = \\ &\quad t_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(x_j, m_{-j}\right)\right], \forall k \in \mathcal{J}, \forall x_j \in L_j \cup H_j, \forall m_{-j} \in M_{-j}^{\mathcal{A}}, \\ &\bar{b}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(x_j\right), m_{-j}\right)\right) = \\ &\quad b_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(x_j, m_{-j}\right)\right], \forall k \in \mathcal{J}, \forall x_j \in L_j \cup H_j, \forall m_{-j} \in M_{-j}^{\mathcal{A}}. \end{aligned}$$

Finally, we extend the continuation equilibrium $\left(\left(c_j'|_{\varphi_j(L_j \cup H_j)}, c_{-j}\right), \bar{s}, \bar{t}\right)$ to a continuation equilibrium $\left(\left(c_j', c_{-j}\right), \bar{s}, \bar{t}\right)$, i.e., we have to extend each principal k's strategy when the agent sends a profile of messages $(m_j, m_{-j}) \notin \varphi_j\left(L_j \cup H_j\right) \times \bar{s}_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$. Fix any $\left[E_j^*, y_j^\circ\right] \in H_j$. Define

$$\begin{aligned} c_j'\left(m_j\right) &= x_j \in L_j \Longrightarrow \\ \bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j, m_{-j}\right)\right) &= t_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(x_j, m_{-j}\right)\right] \\ &= \bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(x_j\right), m_{-j}\right)\right), \forall k \in \mathcal{J} \\ \bar{b}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j, m_{-j}\right)\right) &= b_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(x_j, m_{-j}\right)\right] \\ &= \bar{b}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(x_j\right), m_{-j}\right)\right), \forall k \in \mathcal{J}. \end{aligned}$$

In the situation where the agent's message profile (m_j, m_{-j}) is not in $\varphi_j\left(L_j \cup H_j\right) \times \bar{s}_{-j}\left(\left(c_j', c_{-j}\right), \Theta\right)$, if m_j pins down $x_j \in L_j$ under c_j', principal j associates m_j with $\varphi_j\left(x_j\right)$, or equivalently, upon receiving a message that pins down $x_j \in L_j$ under c_j', principal j forms the same belief as the one induced by receiving $\varphi_j\left(x_j\right)$, and hence, principal j takes the same best reply $\bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(x_j\right), m_{-j}\right)\right)$.

$$\begin{aligned} c_j'\left(m_j\right) &= E_j^* \Longrightarrow \\ \bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j, m_{-j}\right)\right) &= t_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(\left[E_j^*, y_j^\circ\right], m_{-j}\right)\right] \\ &= \bar{t}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(\left[E^*, y_j^\circ\right]\right), m_{-j}\right)\right), \forall k \in \mathcal{J}, \\ \bar{b}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(m_j, m_{-j}\right)\right) &= b_k\left[\Gamma_k\left(\widehat{c}_j', c_{-j}\right), \Psi_k\left(\left[E^*, y_j^\circ\right], m_{-j}\right)\right] \\ &= \bar{b}_k\left(\Gamma_k\left(c_j', c_{-j}\right), \Psi_k\left(\varphi_j\left(\left[E^*, y_j^\circ\right]\right), m_{-j}\right)\right), \forall k \in \mathcal{J}. \end{aligned}$$

In the situation where the agent's message profile (m_j, m_{-j}) is not in $\varphi_j(L_j \cup H_j) \times \bar{s}_{-j}\left(\left(c'_j, c_{-j}\right), \Theta\right)$, if m_j pins down E^*_j under c'_j, principal j associates m_j with $\varphi_j\left(\left[E^*_j, y^\circ_j\right]\right)$, or equivalently, upon receiving a message that pins down E^*_j, principal j forms the same belief as the one induced by receiving $\varphi_j\left(\left[E^*_j, y^\circ_j\right]\right)$, and hence, principal j takes the same best reply $\bar{t}_k\left(\Gamma_k\left(c'_j, c_{-j}\right), \Psi_k\left(\varphi_j\left(\left[E^*_j, y^\circ_j\right]\right), m_{-j}\right)\right)$. As a result, these extra messages bring no harm to the agent's incentive compatibility and principals' incentive compatibility, and therefore, $\left(\left(c'_j, c_{-j}\right), \bar{s}, \bar{t}\right)$ is a continuation equilibrium, and

$$z^{\left(\left(c'_j, c_{-j}\right), \bar{s}, \bar{t}\right)} = z^{\left(\left(\hat{c}'_j, c_{-j}\right), s, t\right)},$$

i.e., c'_j is not a profitable deviation for j. Therefore, $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{F}\right]}}$. ■

On line Appendix B. Proof of Proposition 4

Proof of $Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{F}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{R}, C^{F}\right]}}$

Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{R}, C^{F}\right]}$. We aim to construct an equilibrium $\left[\bar{c}, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle \mathcal{A}, \Gamma, \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{F}\right]}$ such that $z^{\left[\bar{c}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$. We show this by two steps.

For the first step, since $c_j : K_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ is in C^R_j for each $j \in \mathcal{J}$, we have that $K_j \subset M^R_j$ and

$$c_j\left(D_j\right) = \phi(D_j), \forall D_j \in K_j.$$

Fix any injective function $\psi_j : K_j \longrightarrow M^{\mathcal{A}}_j$. That is, for each $D_j \in K_j$, each $\psi_j\left(D_j\right)$ fully represents D_j. We now re-label each $D_j \in K_j$ to $\psi_j\left(D_j\right) \in \psi_j\left(K_j\right)$, and replicate c_j to $\bar{c}_j|_{\psi_j(K_j)} : \psi_j\left(K_j\right) \longrightarrow \mathcal{A}_j$ such that

$$\bar{c}_j\left[\psi_j\left(D_j\right)\right] = \phi(D_j), \forall D_j \in K_j.$$

Similarly, we replicate (s, t) to $\left(\bar{s}, \bar{t}\right)$ subject to such re-labeling. Specifically, $\bar{s}$ is defined as follows.

$$\bar{s}_j\left(\bar{c}, \theta\right) = \psi_j\left[s_j\left(c, \theta\right)\right], \forall j \in \mathcal{J}, \forall \theta \in \Theta. \tag{36}$$

i.e., when $\bar{c}$ is a profile of contracts in $C^{\mathcal{A}}$ that principals offer, the agent of type θ would send the message $\psi_j\left[s_j\left(c, \theta\right)\right]$ to principal j, which results in the menu $\bar{c}_j\left[\psi_j\left(s_j\left(c, \theta\right)\right)\right] =$

$\phi(s_j(c,\theta))$ that the agent would choose for principal j under c_j. If j deviates to $\widehat{c}'_j \in C^F_j$, we have

$$\bar{s}_j\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right) = s_j\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right), \forall \theta \in \Theta, \tag{37}$$

$$\bar{s}_k\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right) = \psi_k\left[s_k\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right)\right], \forall k \neq j, \forall \theta \in \Theta, \tag{38}$$

that is, $\bar{s}$ follows s.

Rigorously, for each $j \in \mathcal{J}$, principal j's action choice strategy $\bar{t}_j$ and her belief $\bar{b}_j$ are defined as follows. Let $\bar{c} = (\bar{c}_j)_{j\in\mathcal{J}}$, $D = (D_j)_{j\in\mathcal{J}}$, $\psi(D) = \left(\psi_j(D_j)\right)_{j\in\mathcal{J}}$, and $K = (K_j)_{j\in\mathcal{J}}$. If every j does not deviate from $\bar{c}_j$, we have

$$\begin{aligned}\bar{t}_j\left(\Gamma_j(\bar{c}), \Psi_j(\psi(A))\right) &= t_j\left(\Gamma_j(c), \Psi_j(A)\right), \forall D \in K. \\ \bar{b}_j\left(\Gamma_j(\bar{c}), \Psi_j(\psi(A))\right) &= b_j\left(\Gamma_j(c), \Psi_j(A)\right), \forall D \in K\end{aligned} \tag{39}$$

i.e., j chooses the action from $\bar{c}_j\left[\psi_j(D_j)\right] = \phi(D_j)$ that she would choose when the agent chooses D_j under c_j. Because the belief is preserved, principal j's action choice is optimal. Further, the belief $\bar{b}_j$ above satisfies the requirement regarding Bayes' rule because b_j satisfies it. If j unilaterally deviates to $\left[\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \setminus \{\varnothing\}\right] \in C^F_j$, we have that for all $k \in \mathcal{J}$

$$\begin{aligned}\bar{t}_k\left(\Gamma_k\left(\widehat{c}'_j, \bar{c}_{-j}\right), \Psi_k\left(m_j, \psi_{-j}(D_{-j})\right)\right) &= t_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), \Psi_k\left(x_j, D_{-j}\right)\right), \\ \forall\left(x_j, D_{-j}\right) &\in \left(L_j \cup H_j\right) \times K_{-j}, \\ \bar{b}_k\left(\Gamma_k\left(\widehat{c}'_j, \bar{c}_{-j}\right), \Psi_k\left(m_j, \psi_{-j}(D_{-j})\right)\right) &= b_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), \Psi_k\left(x_j, D_{-j}\right)\right), \\ \forall\left(x_j, D_{-j}\right) &\in \left(L_j \cup H_j\right) \times K_{-j}\end{aligned}$$

i.e., $\bar{t}_k$ replicates t_k, and $\bar{b}_k$ also replicates b_k for all $k \in \mathcal{J}$. Because $\bar{b}_k$ above satisfies the requirement regarding Bayes' rule because b_k satisfies it.

That is, $\left[\left(\bar{c}_j|_{\psi_j(K_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ replicates (c, s, t) subject to re-labeling each $D_j \in K_j$ to $\psi_j(D_j) \in \psi_j(K_j)$. Clearly, the former inherits incentive compatibility of the latter, and hence, $\left[\left(\bar{c}_j|_{\psi_j(K_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ is an equilibrium subject any principal j's deviation to contracts in C^F_j. Therefore, we have

$$z^{\left[\left(\bar{c}_j|_{\psi_j(K_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}. \tag{40}$$

In Step 2, we extend each $\bar{c}_j|_{\psi_j(K_j)}$ to $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$. Fix an arbitrary $D_j^* \in K_j$. Define

$$\bar{c}_j \left[\psi_j \left(D_j\right)\right] = \phi(D_j), \forall D_j \in K_j,$$
$$\bar{c}_j \left[m_j\right] = D_j^*, \forall m_j \in M_j^{\mathcal{A}} \diagdown \psi_j \left(K_j\right),$$

i.e., $\bar{c}_j$ matches $\bar{c}_j|_{\psi_j(K_j)}$ on the sub-domain $\psi_j \left(K_j\right)$, and when the agent sends a message $m_j \in M_j^{\mathcal{A}} \diagdown \psi_j \left(K_j\right)$, it is equivalent to choose $D_j^* \in K_j$. We also need to extend $\bar{t}_j$ and $\bar{b}_j$when principal chooses $\bar{c}_j$ (rather than $\bar{c}_j|_{\psi_j(K_j)}$) as follows. For all $j \in \mathcal{J}$ and all $m_j \in M_j^{\mathcal{A}}$, define $\psi_j^{-1}\left(m_j\right)$ as

$$\psi_j^{-1}\left(m_j\right) = \begin{cases} D_j & \text{if } \exists D_j \text{ s.t. } m_j = \psi_j(D_j) \\ D_j^* & \begin{array}{c}\text{otherwise, i.e.,} \\ m_j \in M_j^{\mathcal{A}} \backslash \psi_j(K_j)\end{array} \end{cases}$$

Let $\psi^{-1}\left(m\right) = \left(\psi_j^{-1}\left(m_j\right)\right)_{j \in \mathcal{J}}$ and $\psi_{-k}^{-1}\left(m_{-k}\right) = \left(\psi_k^{-1}\left(m_k\right)\right)_{k \neq j}$. Given the extension $\bar{c}_j : M_j^{\mathcal{A}} \longrightarrow 2^{Y_j} \diagdown \{\varnothing\}$ for all $j \in \mathcal{J}$, principal j's action choice and **his** belief upon no principal's deviation follows

$$\bar{t}_j\left(\Gamma_j\left(\bar{c}\right), \Psi_j\left(m\right)\right) = t_j\left(\Gamma_j\left(c\right), \Psi_j\left(\psi^{-1}\left(m\right)\right)\right), \forall m \in M^{\mathcal{A}},$$
$$\bar{b}_j\left(\Gamma_j\left(\bar{c}\right), \Psi_j\left(m\right)\right) = b_j\left(\Gamma_j\left(c\right), \Psi_j\left(\psi^{-1}\left(m\right)\right)\right), \forall m \in M^{\mathcal{A}},$$

Upon principal j's deviation to $\left[\widehat{c}_j' : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}\right] \in C_j^F$, every principal k's action choice and her belief follows

$$\begin{aligned} \bar{t}_k\left(\Gamma_k\left(\widehat{c}_j', \bar{c}_{-j}\right), \Psi_j\left(x_j, m_{-j}\right)\right) &= \\ t_k\left(\Gamma_j\left(\widehat{c}_j', c_{-j}\right), \Psi_j\left(x_j, \psi_{-j}^{-1}\left(m_{-j}\right),\right)\right), \forall m_{-j} &\in M_{-j}^{\mathcal{A}}, \ \forall x_j \in L_j \cup H_j, \\ \bar{b}_k\left(\Gamma_k\left(\widehat{c}_j', \bar{c}_{-j}\right), \Psi_j\left(x_j, m_{-j}\right)\right) &= \\ b_k\left(\Gamma_j\left(\widehat{c}_j', c_{-j}\right), \Psi_j\left(x_j, \psi_{-j}^{-1}\left(m_{-j}\right),\right)\right), \forall m_{-j} &\in M_{-j}^{\mathcal{A}}, \ \forall x_j \in L_j \cup H_j, \end{aligned}$$

The extensions do not change the incentive compatibility of $\bar{s}$ for the agent: under $\bar{c}_k|_{\psi_k(K_k)}$, choosing $\psi_k\left(D_k^*\right)$ is not a profitable deviation for the agent at any continuation equilibrium, and hence, under $\bar{c}_k$, choosing $m_k \in M_k^{\mathcal{A}} \diagdown \psi_k\left(K_k\right)$ will not be a profitable deviation for the agent in any continuation equilibrium either, because it is equivalent to choosing $\psi_k\left(D_k^*\right)$ whether or not there is a principal who deviates to a menu-of-menu-with-full-recommendation contract. Furthermore, given the agent's communication strategy $\bar{s}$, the extension of $\bar{t}_k$ is also the best reply for principal k given the extension of $\bar{b}_k$ for all $k \in \mathcal{J}$.

Therefore, $\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$ is a $\left[C^{\mathcal{A}}, C^F\right]$-equilibrium that replicates $\left[\left(\bar{c}_j|_{\psi_j(K_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]$:

$$z^{\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]} = z^{\left[\left(\bar{c}_j|_{\psi_j(K_j)}\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]}, \tag{41}$$

$$\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}. \tag{42}$$

(36) and (41) imply

$$z^{\left[\left(\bar{c}_j\right)_{j\in\mathcal{J}}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}. \tag{43}$$

Therefore, (42) and (43) imply $Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}} \supset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^R, C^F\right]}}$. ■

Proof of $Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^R, C^F\right]}}$

Fix any $(c,s,t) \in \mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}$, and we aim to construct an equilibrium $\left[\bar{c}, \bar{s}, \bar{t}\right] \in \mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^R, C^F\right]}$ such that $z^{\left[\bar{c}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$.

We first define $\bar{c}_j \in C_j^R$ for each principal j. Define

$$K_j \equiv \left\{ \begin{array}{c} \left[c_j\left(m_j\right), t_j\left[\Gamma_j\left(c_{-k}, \widehat{c}_k'\right), \Psi_j\left(m_{-k}, x_k\right)\right]\right] : \\ \forall k \in \mathcal{J}, \forall \left[\widehat{c}_k' : L_k \cup H_k \rightarrow 2^{Y_j} \diagdown \{\varnothing\}\right] \in C_k^F, \forall x_k \in L_k \cup H_k, \forall m_{-k} \in M_{-k}^{\mathcal{A}} \end{array} \right\},$$

$\bar{c}_j : K_j \rightarrow 2^{Y_j} \diagdown \{\varnothing\}$ is a menu-of-menu-with-recommendation contract such that $\bar{c}_j(D_j) = \phi(D_j)$ for all $D_j \in K_j$

Case of $\left\langle \Gamma^{public}, \Psi^{private} \right\rangle$

$\bar{s}$ will replicate s subject to $\bar{c}$ replicating c. Fix any $j \in \mathcal{J}$. If no principal deviates, we have

$$\bar{s}_j\left(\bar{c}, \theta\right) = \left[c_j\left(s_j\left(c, \theta\right)\right), t_j\left[c, s_j\left(c, \theta\right)\right]\right], \forall j \in \mathcal{J}, \forall \theta \in \Theta,$$

If principal j deviates to $\widehat{c}_j' \in C_j^F$, we have

$$\begin{aligned}
\bar{s}_k\left(\left(\widehat{c}_j', \bar{c}_{-j}\right), \theta\right) &= \left[c_k\left(s_k\left(\left(\widehat{c}_j', \bar{c}_{-j}\right), \theta\right)\right), t_k\left[\left(\widehat{c}_j', \bar{c}_{-j}\right), s_k\left(\left(\widehat{c}_j', \bar{c}_{-j}\right), \theta\right)\right]\right], \forall k \neq j, \forall \theta \in \Theta. \\
\bar{s}_j\left(\left(\widehat{c}_j', \bar{c}_{-j}\right), \theta\right) &= s_j\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right), \forall \theta \in \Theta
\end{aligned}$$

Similarly, $\bar{t}$ replicates t (and $\bar{b}$ replicates b). First, consider the case where no principal deviates from $\bar{c}$. For any D_j in the domain K_j of $\bar{c}_j$, define the following sets:

$$\begin{aligned}
\eta_j\left(D_j\right) &\equiv \left\{s_j\left(c, \theta\right) \in M_j^{\mathcal{A}} : \left[c_j\left(s_j\left(c, \theta\right)\right), t_j\left(c, s_j\left(c, \theta\right)\right)\right] = D_j\right\}, \\
\lambda_j\left(D_j\right) &\equiv \left\{\theta \in \Theta : \left[c_j\left(s_j\left(c, \theta\right)\right), t_j\left(c, s_j\left(c, \theta\right)\right)\right] = D_j\right\}.
\end{aligned}$$

Note that all messages in $\eta_j\left(D_j\right)$ induces the same menu with a recommendation $D_j = \left[E_j, y_j\right] = \left[c_j\left(s_j\left(c, \theta\right)\right), t_j\left(c, s_j\left(c, \theta\right)\right)\right]$. Let $\overline{\eta}_j\left(D_j\right)$ be an arbitrary message in $\eta_j\left(D_j\right)$ if $\eta_j\left(D_j\right) \neq \varnothing$. For the case where $\eta_j\left(D_j\right) = \varnothing$, let $\eta_j^*\left(D_j\right)$ be an arbitrary message in $M_j^{\mathcal{A}}$ such that $\left[c_j\left(\eta_j^*\left(D_j\right)\right), t_j\left(c, \eta_j^*\left(D_j\right)\right)\right] = D_j$. If no principal deviates to an alternative contract, principal j's action choice follows: For all $D_j \in K_j$

$$\bar{t}_j\left(\bar{c}, D_j\right) = \begin{cases} t_j\left(c, \overline{\eta}_j\left(D_j\right)\right) & \text{if } \eta_j\left(D_j\right) \neq \varnothing \\ t_j\left(c, \eta_j^*\left(D_j\right)\right) & \text{otherwise} \end{cases}$$

$$\bar{b}_j\left(\bar{c}, D_j\right) = \begin{cases} \dfrac{\sum_{\theta \in \lambda_j\left(D_j\right)} p(\theta) b_j\left(c, s_j(c, \theta)\right)}{\sum_{\theta' \in \lambda_j\left(D_j\right)} p\left(\theta'\right)} & \text{if } \eta_j^s\left(D_j\right) \neq \varnothing \\ b_j\left(c, \eta_j^*\left(D_j\right)\right) & \text{otherwise} \end{cases}$$

Because b_j above satisfies the requirement regarding Bayes' rule, $\bar{b}_j$ above also satisfies it. Let $D_j = \left[E_j, y_j\right]$. Moreover, if $\eta_j\left(D_j\right) \neq \varnothing$, principal j optimally chooses the same action y_j when she receives any message m_j in $\eta_j\left(D_j\right)$ from E_j given her belief $b_j\left(c, m_j\right)$. Because principal j's utility function satisfies the expected utility property, it implies that given the weighted sum of such beliefs, principal j's optimal choice of action is also y_j. In addition, if $\eta_j\left(D_j\right) = \varnothing$, it is clear that $\bar{t}_j\left(\bar{c}, D_j\right) = t_j\left(c, \eta_j^*\left(D_j\right)\right)$ is principal j's optimal action choice given her belief $\bar{b}_j\left(\bar{c}, D_j\right) = b_j\left(c, \eta_j^*\left(D_j\right)\right)$. Further, $z^{\left[\bar{c}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$.

For all $k \in \mathcal{J}$, all D_k in the domain K_k of $\bar{c}_k$, all $j \neq k$, and all $\left[\widehat{c}_j' : L_j \cup H_j \to 2^{Y_j} \setminus \{\varnothing\}\right] \in C_j^F$, define the following sets:

$$\begin{aligned}
\kappa_k\left(D_k, \widehat{c}_j'\right) &\equiv \left\{\begin{array}{c} s_k\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right) \in M_k^{\mathcal{A}} : \\ \left[c_k\left(s_k\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right)\right), t_k\left(\left(\widehat{c}_j', c_{-j}\right), s_k\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right)\right)\right] = D_k \end{array}\right\}, \\
\tau_k\left(D_j, \widehat{c}_j'\right) &\equiv \left\{\theta \in \Theta : \left[c_k\left(s_k\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right)\right), t_k\left(\left(\widehat{c}_j', c_{-j}\right), s_k\left(\left(\widehat{c}_j', c_{-j}\right), \theta\right)\right)\right] = D_k\right\}.
\end{aligned}$$

If $\kappa_k\left(D_k,\widehat{c}_j'\right)\neq\varnothing$, let $\overline{\kappa}_k\left(D_k,\widehat{c}_j'\right)$ be an arbitrary message in $\kappa_k\left(D_k,\widehat{c}_j'\right)$. Otherwise, let $\kappa_k^*\left(D_k,\widehat{c}_j'\right)$ be an arbitrary message in $M_j^{\mathcal{A}}$ such that $\left[c_k\left(\kappa_k^*\left(D_k,\widehat{c}_j'\right)\right),t_k\left(c,\kappa_k^*\left(D_k,\widehat{c}_j'\right)\right)\right]=D_k$. If j deviates to $\left[\widehat{c}_j':L_j\cup H_j\to 2^{Y_j}\diagdown\{\varnothing\}\right]\in C_j^F$, we have

$$\begin{aligned}
\overline{t}_j\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),m_j\right) &= t_j\left(\left(\widehat{c}_j',c_{-j}\right),x_j\right),\forall x_j\in L_j\cup H_j,\\
\overline{b}_j\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),m_j\right) &= b_j\left(\left(\widehat{c}_j',c_{-j}\right),x_j\right),\forall x_j\in L_j\cup H_j,\\
\overline{t}_k\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),D_k\right) &= \begin{cases} t_k\left(\left(\widehat{c}_j',c_{-j}\right),\overline{\kappa}_k\left(D_k,\widehat{c}_j'\right)\right) & \text{if } \kappa_k\left(D_k,\widehat{c}_j'\right)\neq\varnothing\\ t_k\left(\left(\widehat{c}_j',c_{-j}\right),\kappa_k^*\left(D_k,\widehat{c}_j'\right)\right) & \text{otherwise}\end{cases},\forall D_k\in K_k,\\
\overline{b}_k\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),D_k\right) &= \begin{cases} \dfrac{\sum_{\theta\in\tau_j\left(D_j,\widehat{c}_j'\right)}p(\theta)b_j\left(c,s_j(c,\theta)\right)}{\sum_{\theta'\in\tau_j\left(D_j,\widehat{c}_j'\right)}p\left(\theta'\right)} & \text{if } \kappa_k\left(D_k,\widehat{c}_j'\right)\neq\varnothing\\ b_j\left(c,\kappa_k^*\left(D_k,\widehat{c}_j'\right)\right) & \text{otherwise}\end{cases},\ \forall D_k\in K_k.
\end{aligned}$$

Note that $\overline{b}_j\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),m_j\right)$ satisfies the requirement of Bayes' rule because $b_j\left(\left(\widehat{c}_j',c_{-j}\right),m_j\right)$ satisfies it. Given the same belief, $\overline{t}_j\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),m_j\right)=t_j\left(\left(\widehat{c}_j',c_{-j}\right),m_j\right)$ is the optimal action choice for principal j. Note that Bayes' rule does not apply to $\overline{b}_k\left(\left(\widehat{c}_j',\overline{c}_{-j}\right),D_k\right)$ because it is off the path following principal j's deviation to $\widehat{c}_j'$. However, if $\kappa_k\left(D_k,\widehat{c}_j'\right)\neq\varnothing$, principal k optimally chooses the same action y_k from E_k in $D_k=[E_k,y_k]$ when she receives any message m_k in $\kappa_k\left(D_k,\widehat{c}_j'\right)$ given her belief $b_j\left(\left(\widehat{c}_j',c_{-j}\right),m_j\right)$. Because principal j's utility function satisfies the expected utility property, it implies that given the weighted sum of such beliefs, principal k's optimal choice of action is also y_k. Therefore, $\left[\overline{c},\overline{s},\overline{t}\right]$ replicates the allocation of (c,s,t) regarding any possible unilateral deviations by principals i.e., $z^{\left[\left(\widehat{c}_j',\overline{c}_{-j}\right),\overline{s},\overline{t}\right]}=z^{\left(\left(\widehat{c}_j',c_{-j}\right),s,t\right)}$, $\forall j$, $\forall\widehat{c}_j'\in C_j^F$. Hence, $\left[\overline{c},\overline{s},\overline{t}\right]$ inherits the incentive compatibility of principals and the agent from that of (c,s,t) on and off the path following a player's unilateral deviation. As a result, $\left[\overline{c},\overline{s},\overline{t}\right]\in Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^R,C^F\right]}}$. Because $z^{\left[\overline{c},\overline{s},\overline{t}\right]}=z^{(c,s,t)}$, we have that $Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^{\mathcal{A}},C^F\right]}}\subset Z^{\mathcal{E}^{\langle\mathcal{A},\Gamma,\Psi\rangle\text{-}\left[C^R,C^F\right]}}$. ■

Cases of $\left\langle\Gamma^{private},\ \Psi^{public}\right\rangle$ and $\left\langle\Gamma^{public},\ \Psi^{public}\right\rangle$

We prove both cases of $\left\langle\Gamma^{private},\Psi^{public}\right\rangle$ and $\left\langle\Gamma^{public},\Psi^{public}\right\rangle$ together. Fix $\Gamma\in\left\{\Gamma^{private},\Gamma^{public}\right\}$. $\overline{s}$ will replicate s subject to $\overline{c}$ replicating c. Fix any $j\in\mathcal{J}$. If no principal deviates, we have

$$\overline{s}_j\left(\overline{c},\theta\right)=\left[c_j\left(s_j\left(c,\theta\right)\right),t_j\left[\Gamma_j\left(c\right),s\left(c,\theta\right)\right]\right],\forall j\in\mathcal{J},\forall\theta\in\Theta.$$

If principal j deviates to $\widehat{c}'_j \in C_j^F$, we have

$$\begin{aligned}
\bar{s}_k\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right) &= \left[c_k\left(s_k\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right)\right), t_k\left[\Gamma_k\left(\widehat{c}'_j, \bar{c}_{-j}\right), s\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right)\right]\right], \forall k \neq j, \forall \theta \in \Theta. \\
\bar{s}_j\left(\left(\widehat{c}'_j, \bar{c}_{-j}\right), \theta\right) &= s_j\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right), \forall \theta \in \Theta
\end{aligned}$$

Similarly, $\bar{t}$ replicates t (and $\bar{b}$ replicates b). First consider the case where no principal deviates from $\bar{c}$. Let $D = (D_1, \ldots, D_J)$ given any D_j in the domain K_j of $\bar{c}_j$ for all $j \in \mathcal{J}$ define the following sets:

$$\begin{aligned}
\eta(D) &\equiv \left\{s(c,\theta) \in M^{\mathcal{A}} : \left[c_k\left(s_k(c,\theta)\right), t_k\left(\Gamma_k(c), s(c,\theta)\right)\right] = D_k, \forall k \in \mathcal{J}\right\}, \\
\lambda(D) &\equiv \left\{\theta \in \Theta : \left[c_k\left(s_k(c,\theta)\right), t_k\left(\Gamma_k(c), s(c,\theta)\right)\right] = D_k, \forall k \in \mathcal{J}\right\}.
\end{aligned}$$

Note that all message profile in $\eta(D)$ induce the same menu with a recommendation $D_j = \left[E_j, y_j\right] = \left[c_j\left(s_j(c,\theta)\right), t_j\left(\Gamma_j(c), s(c,\theta)\right)\right]$ for each principal j, leading principal j to choose the same action y_j from E_j. Let $\overline{\eta}(D)$ be an arbitrary message profile in $\eta(D)$ if $\eta(D) \neq \varnothing$. For the case where $\eta(D) = \varnothing$, let $\eta^*(D) = \left(m_1^*, \ldots, m_J^*\right)$ be an arbitrary message profile in $M^{\mathcal{A}}$ such that $\left[c_j\left(m_j^*\right), t_j\left(\Gamma_j(c), \eta^*(D)\right)\right] = D_j$.

If no principal deviates to an alternative contract, principal j's action choice follows: For all $D_j \in K_j$

$$\begin{aligned}
\bar{t}_j\left(\Gamma_j(\bar{c}), D\right) &= \begin{cases} t_j\left(\Gamma_j(c), \overline{\eta}(D)\right) & \text{if } \eta(D) \neq \varnothing \\ t_j\left(\Gamma_j(c), \eta^*(D)\right) & \text{otherwise} \end{cases}, \\
\bar{b}_j\left(\Gamma_j(\bar{c}), D\right) &= \begin{cases} \frac{\sum_{\theta \in \lambda(D)} p(\theta) b_j\left(\Gamma_j(c), s(c,\theta)\right)}{\sum_{\theta' \in \lambda(D)} p(\theta')} & \text{if } \eta(D) \neq \varnothing \\ b_j\left(\Gamma_j(c), \eta^*(D)\right) & \text{otherwise} \end{cases}
\end{aligned}$$

$\bar{b}_j$ above satisfies the requirement regarding Bayes' rule given that b_j satisfies it. We have $z^{[\bar{c}, \bar{s}, \bar{t}]} = z^{(c,s,t)}$ and the optimality of principals' action choices are preserved.

For all $j \in \mathcal{J}$, suppose that principal j deviates to $\left[\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \setminus \{\varnothing\}\right] \in C_j^F$. Given $\left[\widehat{c}'_j : L_j \cup H_j \to 2^{Y_j} \setminus \{\varnothing\}\right]$, every $x_j \in L_j \cup H_j$, every $k \neq j$, and every D_k in the domain of $\overline{c}_k$, define the following sets:

$$\kappa\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right) \equiv \left\{\begin{array}{c} s\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\in M^{\mathcal{A}}: \\ \left[c_k\left(s_k\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\right),t_k\left(\Gamma_k\left(\widehat{c}'_j,c_{-j}\right),s\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\right)\right]=D_k,\ \forall k\neq j, \\ s_j\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)=x_j \end{array}\right\},$$

$$\tau\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right) \equiv \left\{\begin{array}{c} \theta\in\Theta: \\ \left[c_k\left(s_k\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\right),t_k\left(\Gamma_k\left(\widehat{c}'_j,c_{-j}\right),s\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\right)\right]=D_k,\ \forall k\neq j, \\ s_j\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)=x_j \end{array}\right\}.$$

If $\kappa\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\neq\varnothing$, let $\overline{\kappa}\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)$ be an arbitrary message profile in $\kappa\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)$. Otherwise, let $\kappa^*\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)=\left(x_j,(m^*_k)_{k\neq j}\right)$ be an arbitrary message profile in $M^{\mathcal{A}}$ such that

$$\left[c_k\left(m^*_k\right),t_k\left(\Gamma_k\left(\widehat{c}'_j,c_{-j}\right),\kappa^*\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\right)\right]=D_k,\forall k\neq j.$$

When principal j deviates to $\left[\widehat{c}'_j: L_j\cup H_j\to 2^{Y_j}\diagdown\{\varnothing\}\right]\in C^F_j$, her action choice strategy and belief follows that for all $x_j\in L_j\cup H_j$, all $k\neq j$, and all $D_k\in K_k$,

$$\overline{t}_j\left(\Gamma_j\left(\widehat{c}'_j,\overline{c}_{-j}\right),\left(x_j,(D_k)_{k\neq j}\right)\right)=$$
$$\left\{\begin{array}{ll} t_j\left(\left(\widehat{c}'_j,c_{-j}\right),\overline{\kappa}^s\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\right) & \text{if } \kappa^s\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\neq\varnothing \\ t_j\left(\left(\widehat{c}'_j,c_{-j}\right),\kappa^*\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\right) & \text{otherwise} \end{array}\right.$$
$$\overline{b}_j\left(\Gamma_j\left(\widehat{c}'_j,\overline{c}_{-j}\right),\left(x_j,(D_k)_{k\neq j}\right)\right)=$$
$$\left\{\begin{array}{ll} \dfrac{\sum_{\theta\in\tau^s\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)}p(\theta)b_j\left(\Gamma_j\left(\widehat{c}'_j,c_{-j}\right),s\left(\left(\widehat{c}'_j,c_{-j}\right),\theta\right)\right)}{\sum_{\theta'\in\tau^s\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)}p\left(\theta'\right)} & \text{if } \kappa^s\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\neq\varnothing \\ b_j\left(\Gamma_j\left(\widehat{c}'_j,c_{-j}\right),\kappa^*\left(x_j,(D_k)_{k\neq j},\widehat{c}'_j\right)\right) & \text{otherwise} \end{array}\right.,$$

The belief $\overline{b}_j$ above satisfies the requirement regarding Bayes' rule given that b_j satisfies it. When principal j deviates to $\left[\widehat{c}'_j: L_j\cup H_j\to 2^{Y_j}\diagdown\{\varnothing\}\right]\in C^F_j$,non-deviating principal k's action choice strategy and belief follow that for all $x_j\in L_j\cup H_j$, all $k\neq j$, and all $D_k\in K_k$,

$$\bar{t}_k\left(\Gamma_k\left(\widehat{c}'_j, \bar{c}_{-j}\right), \left(x_j, (D_k)_{k\neq j}\right)\right) =$$
$$\begin{cases} t_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), \bar{\kappa}\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right)\right) & \text{if } \kappa\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right) \neq \varnothing \\ t_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), \kappa^*\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right)\right) & \text{otherwise} \end{cases},$$
$$\bar{b}_k\left(\Gamma_k\left(\widehat{c}'_j, \bar{c}_{-j}\right), \left(x_j, (D_k)_{k\neq j}\right)\right) =$$
$$\begin{cases} \frac{\sum_{\theta\in\tau\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right)} p(\theta) b_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), s\left(\left(\widehat{c}'_j, c_{-j}\right), \theta\right)\right)}{\sum_{\theta'\in\tau^s\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right)} p\left(\theta'\right)} & \text{if } \kappa\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right) \neq \varnothing \\ b_k\left(\Gamma_k\left(\widehat{c}'_j, c_{-j}\right), \kappa^*\left(x_j, (D_k)_{k\neq j}, \widehat{c}'_j\right)\right) & \text{otherwise} \end{cases}.$$

The belief $\bar{b}_k$ above satisfies the requirement regarding Bayes' rule given that b_k satisfies it.

$\left[\bar{c}, \bar{s}, \bar{t}\right]$ replicates the allocation of (c, s, t) regarding any possible unilateral deviations by principals i.e., $z^{\left[\left(\widehat{c}'_j, \bar{c}_{-j}\right), \bar{s}, \bar{t}\right]} = z^{\left(\left(\widehat{c}'_j, c_{-j}\right), s, t\right)}$, $\forall j$, $\forall \widehat{c}'_j \in C^F_j$. Hence, $\left[\bar{c}, \bar{s}, \bar{t}\right]$ inherits the incentive compatibility of principals and the agent from that of (c, s, t) on and off the path following a player's unilateral deviation. As a result, $\left[\bar{c}, \bar{s}, \bar{t}\right] \in Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^R, C^F\right]}}$. Because $z^{\left[\bar{c}, \bar{s}, \bar{t}\right]} = z^{(c,s,t)}$, we have that $Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^F\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A}, \Gamma, \Psi\rangle\text{-}\left[C^R, C^F\right]}}$. ■