Online Appendix to "Common Agency with Non-Delegation or Imperfect Commitment"

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In this note, we follow the same notation as in Han and Xiong (2022). In Section 1, we extend our model to the one with the agent's effort. In Section 2, we consider mixed strategies. In Section 3, we prove Theorem 5 in Han and Xiong (2022).

1 Agent's effort

Suppose that there is a Stage 4 in the game, in which the agent chooses his effort after observing principals' actions at Stage 3. Let X be the set of all efforts that the agent can choose, and principals' and the agent's utility depends on the effort. Specifically, each principal j's utility function is $v_j^e : X \times Y \times \Theta \to \mathbb{R}$. The agent's utility function is $u^e : X \times Y \times \Theta \to \mathbb{R}$.

At Stage 4, the agent chooses his effort in X conditional on his type, principals' contracts, his messages and principals' action choices. Let $e : C \times M \times Y \times \Theta \to X$ denote the agent's effort strategy. Define

$$\Pi \equiv \left\{ e : C \times M \times Y \times \Theta \to X : \begin{array}{c} e\left(c, m, x, y\right) \in \arg\max_{x \in X} u^e\left(x, y, \theta\right), \\ \forall \left(c, m, y, \theta\right) \in C \times M \times Y \times \Theta \end{array} \right\}.$$

Thus, in any equilibrium, the agent's behavior strategy at Stage 4 must be a function $e^* : C \times M \times Y \times \Theta \to X$ in Π. Fix any $e^* \in \Pi$, consider any equilibrium in which the agent adopt

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e^* at Stage 4.—Since this common knowledge among all players, by using an argument as in backward induction, we can reduce the 4-stage game to a 3-stage game with e^* embedded to players' utility functions. Given a strategy and type profile (c, s, t, θ), the agent's utility function $U(c, s, t, \theta)$ can be redefined as

$$U(c, s, t, \theta) \equiv u^e\left(e^*\left[c, s(c, \theta), t(\Gamma(c), \Psi(s(c, \theta))), \theta\right], t(\Gamma(c), \Psi(s(c, \theta))), \theta\right),$$

where $s(c, \theta) = [s_k(c, \theta)]_{k \in \mathcal{J}}$ and $t(\Gamma(c), \Psi(s(c, \theta))) = [t_k(\Gamma_k(c), \Psi_k(s(c, \theta)))]_{k \in \mathcal{J}}$. Similarly, give a strategy profile (c, s, t), principal j's expected utility $V_j(c, s, t)$ can be redefined as

$$V_j(c, s, t) \equiv \sum_{\theta \in \Theta} p(\theta)\, v_j^e\left(e^*\left[c, s(c, \theta), t(\Gamma(c), \Psi(s(c, \theta))), \theta\right], t(\Gamma(c), \Psi(s(c, \theta))), \theta\right)$$

Further, given belief b_j, principal j's expected utility conditional on $(\alpha_j, \beta_j) \in \Gamma_j(C) \times \Psi_j(M)$ is

$$V_j\left(t_j, t_{-j} | \alpha_j, \beta_j, b_j\right) \equiv \int\limits_{b_j(\alpha_j, \beta_j)} v_j^e\left(e^*\left[c, m, t(\Gamma(c), \Psi(m)), \theta\right], t(\Gamma(c), \Psi(m)), \theta\right) d(c, m, \theta),$$

where $t(\Gamma(c), \Psi(m)) = [t_k(\Gamma_k(c), \Psi_k(m))]_{k \in \mathcal{J}}$.

As a result, the 3-stage game is the model defined in Section 3.2 in Han and Xiong (2022), and our results imply

$$\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private} \right\rangle \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{P}, C^{F}\right]}},$$

$$\left(\begin{array}{c} \mathcal{A} = \mathcal{A}^{non-delegated} \text{ and} \\ \langle \Gamma,\ \Psi \rangle \in \left\{ \left\langle \Gamma^{public},\ \Psi^{private} \right\rangle, \left\langle \Gamma^{public},\ \Psi^{public} \right\rangle \right\} \end{array} \right) \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{R}, C^{F}\right]}},$$

$$\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{public} \right\rangle \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi \rangle\text{-}\left[C^{R}, C^{F^*}\right]}}.$$

Since this is true for any $e^* \in \Pi$, we conclude that all of the results still hold for the 4-stage game with the agent's effort.

2 Mixed-strategy equilibria

Though we focus on pure-strategy equilibria in our analysis, our results can be easily extended for mixed-strategy equilibria. To accommodate the change, we need to modify the definitions of menus, menu-of-menu-with-recommendation contracts and menu-of-menu-with-full-recommendation contracts.

Our presumption on $C^{\mathcal{A}^{non-delegated}}$ is that $M_j^{\mathcal{A}^{non-delegated}}$ is so general that any equilibrium allocation in $Z^{\mathcal{E}^{\langle \mathcal{A},\, \Gamma,\, \Psi \rangle \text{-} \left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}}$ is robust in the sense that it survives even when a principal deviates to a more complex contract, which is not in $C^{\mathcal{A}^{non-delegated}}$. Given that our interest is mixed-strategy equilibrium allocations, that might not be the case if $\left|M_j^{\mathcal{A}^{non-delegated}}\right| < |\triangle(Y_j)|$. Therefore, we assume that $\left|M_j^{\mathcal{A}^{non-delegated}}\right| \geq |\triangle(Y_j)|$.

New definition of menus: A menu is a contract, $c_j^{\triangle} : E_j \to Y_j$, such that

$$E_j \in 2^{\triangle(Y_j)} \diagdown \{\varnothing\} \text{ and } c_j^{\triangle}(y_j) = y_j,\ \forall y_j \in E_j.$$

While principal j delegates the randomization of her action choice to the agent in a menu contract, she randomizes her action choice in menu-of-menu-with-recommendation contracts and menu-of-menu-with-full-recommendation contracts. Let $C^{P\text{-}\triangle}$ denote the set of such menu profiles.

New definition of menu-of-menu-with-recommendation contracts: For each $j \in \mathcal{J}$, define

$$M_j^{R\text{-}\triangle} \equiv \left\{[E_j, y_j] : E_j \in 2^{Y_j} \diagdown \{\varnothing\} \text{ and } y_j \in \triangle(E_j)\right\}.$$

Then, a menu-of-menu-with-recommendation contract for principal j is a pair of (i) $K_j \subset 2^{M_j^{R\text{-}\triangle}} \diagdown \{\varnothing\}$ and (ii) a function, $c_j^{\triangle} : K_j \to 2^{Y_j} \diagdown \{\varnothing\}$ such that

$$c_j^{\triangle}([E_j, y_j]) = E_j,\ \forall [E_j, y_j] \in K_j.$$

Note that the agent's recommendation to principal j is a mixed action (i.e., we replace M_j^R with $M_j^{R\text{-}\triangle}$). Let $C^{R\text{-}\triangle}$ denote the set of such menu-of-menu-with-recommendation contract profiles.

New definition of menu-of-menu-with-full-recommendation contracts: A menu-of-menu-with-full-recommendation contract for principal j is a function $c_j^{\triangle} : L_j \cup H_j \to 2^{Y_j} \diagdown \{\varnothing\}$ with

$$H_j = \{[E_j, y_j] : y_j \in \triangle(E_j)\} \text{ for some } E_j \in 2^{Y_j} \diagdown \{\varnothing\},$$
$$L_j \subset 2^{Y_j} \diagdown \{E_j\},$$

such that

$$
\begin{aligned}
c_j^{\triangle}\left(E_j'\right) &= E_j', \forall E_j' \in L_j, \\
c_j^{\triangle}\left([E_j, y_j]\right) &= E_j, \forall \left[E_j, y_j\right] \in H_j.
\end{aligned}
$$

Let $C^{F\text{-}\triangle} \equiv \times_{k\in\mathcal{J}} C_k^{F\text{-}\triangle}$ denote the set of such menu-of-menu-with-full-recommendation contract profiles.

Furthermore, define

$$
C^{F^*\text{-}\triangle} \equiv \times_{k\in\mathcal{J}} \left[C_k^{F\text{-}\triangle} \cup \left\{c_k^{y_k} : y_k \in Y_k\right\}\right],
$$

where $c_k^{y_k} : M_k^{R-F} \to 2^{Y_k} \diagdown \{\varnothing\}$ denote the following degenerate contract:

$$
c_k^{y_k}\left(m_k\right) = \left\{y_k\right\}, \forall m_k \in M_k^{R-F}
$$

as defined in Section 7.2 in Han and Xiong (2022).

With the changes above, it is easy to show

$$
\begin{aligned}
&\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{private}\right\rangle \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P\text{-}\triangle}, C^{F\text{-}\triangle}\right]}}, \\
&\left(\begin{array}{c} \mathcal{A} = \mathcal{A}^{non-delegated} \text{ and} \\ \langle \Gamma,\ \Psi\rangle \in \left\{\left\langle \Gamma^{public},\ \Psi^{private}\right\rangle, \left\langle \Gamma^{public},\ \Psi^{public}\right\rangle\right\} \end{array}\right) \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{P\text{-}\triangle}, C^{F\text{-}\triangle}\right]}}, \\
&\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated},\ \Gamma^{private},\ \Psi^{public}\right\rangle \Rightarrow Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}\|}, C^{\mathcal{A}}\right]}} = Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R\text{-}\triangle}, C^{F^*\text{-}\triangle}\right]}}.
\end{aligned}
$$

3 Proof of Theorem 5 in Han and Xiong (2022)

We need the following three results to prove Theorem 5 in Han and Xiong (2022), and their proofs are relegated to Sections 3.1, 3.2 and 3.3.

Proposition 1 *Given* $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public}\right\rangle$, *we have*

$$
Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}}.
$$

Proposition 2 *Given* $\langle \mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public}\right\rangle$, *we have*

$$
Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{F^*}\right]}}.
$$

Proposition 3 *Given* $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public} \right\rangle$, *we have*

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{F^*}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}}.$$

Proof of Theorem 5 in Han and Xiong (2022). Given $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public} \right\rangle$, Propositions 1 and 2 imply

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{F^*}\right]}}. \tag{1}$$

Furthermore, we have

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{F^*}\right]}} \supset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{F^*}\right]}}. \tag{2}$$

where the first "$\supset$" follows from Proposition 3, the second "$\supset$" from $C^{\mathcal{A}} \sqsupset^* C^R$ and Lemma 1 in Han and Xiong (2022). Finally, (1) and (2) imply Theorem 5.■

3.1 Proof of Proposition 1

Fix $\langle \mathcal{A},\ \Gamma,\ \Psi \rangle = \left\langle \mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public} \right\rangle$, and we aim to prove

$$Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}}.$$

Fix any $(c, s, t) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$. We aim to replicate (c, s, t) with $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right) \in \mathcal{E}^{\langle \mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{R}, C^{\mathcal{A}}\right]}$ such that $z^{(c,s,t)} = z^{\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)}$.

Replicating c with $c^{(c,s,t)} \in C^R$:

On the equilibrium path, for each $j \in \mathcal{J}$, offering c_j is equivalent to offering the menu-of-menu-with-recommendation contract, $c_j^{(c,s,t)} : M_j^{(c,s,t)} \longrightarrow Y_j$ with

$$M_j^{(c,s,t)} \equiv \left\{ \left[E_j = c_j\left[s_j\left(c', \theta\right)\right],\ y_j = t_j\left[\Gamma_j\left(c'\right), \Psi_j\left(s\left(c', \theta\right)\right)\right]\right] : \begin{array}{c} c'_j = c_j, \\ \left(c'_{-j}, \theta\right) \in C^{\mathcal{A}}_{-j} \times \Theta \end{array} \right\}, \tag{3}$$

$$c_j^{(c,s,t)}\left[E_j,\ y_j\right] = E_j,\ \forall \left[E_j,\ y_j\right] \in M_j^{(c,s,t)}.$$

Given $\left(\left(c_j, c'_{-j}\right), \theta\right) \in C^{\mathcal{A}} \times \Theta$, if all players follow (s, t), the subset $E_j = c_j\left[s_j\left(\left(c_j, c'_{-j}\right), \theta\right)\right]$ is fixed for j at Stage 2, and j takes the action $y_j = t_j\left[\Gamma_j\left(\left(c_j, c'_{-j}\right)\right), \Psi_j\left(s\left(\left(c_j, c'_{-j}\right), \theta\right)\right)\right]$

at Stage 3. $M_j^{(c,s,t)}$ is the set of all such profiles. Define $c^{(c,s,t)} \equiv \left(c_k^{(c,s,t)}\right)_{k\in\mathcal{J}} \in C^R$, and let

$$\begin{aligned} \widehat{C} &\equiv \times_{k\in\mathcal{J}} \widehat{C}_k \equiv \times_{k\in\mathcal{J}} \left(\left\{c_k^{(c,s,t)}\right\} \cup C_k^{\mathcal{A}}\right) \text{ and} \\ \widehat{M} &\equiv \times_{k\in\mathcal{J}} \widehat{M}_k \equiv \times_{k\in\mathcal{J}} \left(M_k^{(c,s,t)} \cup M_k^{\mathcal{A}}\right) \end{aligned}$$

denote the relevant contract space and the relevant message space respectively in the $\left[C^R, C^{\mathcal{A}}\right]$-game. In the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, $C^{\mathcal{A}}$ and $M^{\mathcal{A}}$ are the relevant contract space and message space, respectively.

Replicating $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)}\right)_{k\in\mathcal{J}}$:

When the agent observes $c_j^{(c,s,t)}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game, he interprets it as c_j in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, due to the replication process above. Also, when the agent observes $c'_j \in C_j^{\mathcal{A}}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game, he interprets it as $c'_j \in C_j^{\mathcal{A}}$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game. To record this interpretation, define $\gamma_j : \widehat{C}_j \longrightarrow C_j^{\mathcal{A}}$ for each $j \in \mathcal{J}$ as

$$\gamma_j\left(\widehat{c}_j\right) \equiv \begin{cases} c_j, & \text{if } \widehat{c}_j = c_j^{(c,s,t)}; \\ \widehat{c}_j, & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}}, \end{cases} \tag{4}$$

and denote $\gamma(\widehat{c}) \equiv \left(\left[\gamma_k\left(\widehat{c}_k\right)\right]_{k\in\mathcal{J}}\right) \in C^{\mathcal{A}}$ for all $\widehat{c} = \left(\widehat{c}_k\right)_{k\in\mathcal{J}} \in \widehat{C}$.

For the agent's strategy in the $\left[C^R, C^{\mathcal{A}}\right]$-game, we replicate $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^{(c,s,t)} \equiv \left(s_k^{(c,s,t)}\right)$ defined as follows. For each $j \in \mathcal{J}$ and all $(\widehat{c}, \theta) \in \widehat{C} \times \Theta$,

$$s_j^{(c,s,t)}(\widehat{c}, \theta) = \begin{cases} s_j\left[\gamma(\widehat{c}),\ \theta\right], & \text{if } \widehat{c}_j \in C_j^{\mathcal{A}}; \\ \left[E_j = c_j\left(s_j\left[\gamma(\widehat{c}),\ \theta\right]\right),\ y_j = t_j\left(\Gamma_j\left(\gamma(\widehat{c})\right), \Psi_j\left[s\left(\gamma(\widehat{c}), \theta\right)\right]\right)\right], & \text{if } \widehat{c}_j = c_j^{(c,s,t)}. \end{cases}$$

I.e., when principals offer $\widehat{c}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game, the agent regards it as $\gamma(\widehat{c})$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game. Given $\gamma(\widehat{c})$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, the agent sends $s_j\left[\gamma(\widehat{c}),\ \theta\right]$ to j at Stage 2, which pins down the subset $c_j\left(s_j\left[\gamma(\widehat{c}),\ \theta\right]\right)$ for j, and j takes the action $t_j\left(\Gamma_j\left(\gamma(\widehat{c})\right), \Psi_j\left[s\left(\gamma(\widehat{c}), \theta\right)\right]\right)$ at Stage 3. Then, in the $\left[C^R, C^{\mathcal{A}}\right]$-game, $s_j^{(c,s,t)}$ replicates s_j: if $\widehat{c}_j \in C_j^{\mathcal{A}}$, the agent sends $s_j\left[\gamma(\widehat{c}),\ \theta\right]$ to j; if $\widehat{c}_j = c_j^{(c,s,t)} \in C_j^R$, the agent chooses the subset $c_j\left(s_j\left[\gamma(\widehat{c}),\ \theta\right]\right)$ with the recommendation $t_j\left(\Gamma_j\left(\gamma(\widehat{c})\right), \Psi_j\left[s\left(\gamma(\widehat{c}), \theta\right)\right]\right)$.

Replicating (b_j, t_j) with $\left(b_j^{(c,s,t)}, t_j^{(c,s,t)}\right)$:

Given $\left\langle \Gamma^{private}, \Psi^{public} \right\rangle$, each principal j observes only $\left(c'_j, m'\right)$. Thus, for notational simplicity, we write $t_j\left(c'_j, m'\right)$ and $b_j\left(c'_j, m'\right)$ for $t_j\left(\Gamma_j\left(c'\right), \Psi_j\left(m'\right)\right)$ and $b_j\left(\Gamma_j\left(c'\right), \Psi_j\left(m'\right)\right)$, respectively. Furthermore, j has a degenerate belief on $C_j \times M$, and hence, we describe only the marginal belief on $C_{-j} \times \Theta$. Consider

$$Q_j \equiv \left\{ (\widehat{c}_j, \widehat{m}) : \widehat{c}_j \in \widehat{C}_j \text{ and } \widehat{m} \in \widehat{M} \right\},$$

$$Q_j^* \equiv \left\{ \left(\widehat{c}_j, s^{(c,s,t)} \left[\left(\widehat{c}_j, c_{-j}^{(c,s,t)} \right), \theta \right] \right) \in Q_j : \widehat{c}_j \in \widehat{C}_j \text{ and } \theta \in \Theta \right\},$$

i.e., in the $\left[C^R, C^{\mathcal{A}}\right]$-game, Q_j is the set of all possible information that principal j may observe before j takes an action at Stage 3; Q_j^* is the subset of information by which j cannot confirm that the other players have deviated.

For each $q_j \in Q_j^*$, fix any $\left(c_j^{q_j}, \theta^{q_j}\right)$ such that

$$q_j = \left(c_j^{q_j}, s^{(c,s,t)} \left[\left(c_j^{q_j}, c_{-j}^{(c,s,t)} \right), \theta^{q_j} \right] \right).$$

For each $q_j = (\widehat{c}_j, \widehat{m}) \in Q_j \diagdown Q_j^*$, fix any $\left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right) \in C_j^{\mathcal{A}} \times M^{\mathcal{A}}$ such that

$$\widehat{c}_j \left(\widehat{m}_j \right) = \widetilde{c}_j^{q_j} \left(\widetilde{m}_j^{q_j} \right).$$

We record this as $\Sigma_j : Q_j \longrightarrow Q_j^{\mathcal{A}}$ such that

$$\Sigma_j (q_j) \equiv \begin{cases} \left[\gamma_j \left(c^{q_j}\right), \ s\left[\left(\gamma_j\left(c^{q_j}\right), c_{-j}\right), \theta^{q_j}\right]\right] & \text{if } q_j \in Q_j^*; \\ \\ \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right) & \text{if } q_j \in Q_j \diagdown Q_j^* \end{cases},$$

$$\text{where } Q_j^{\mathcal{A}} \equiv \left\{ \left(\Gamma_j\left(c'\right), \Psi_j\left(m\right)\right) : \begin{array}{c} c' = \left(c'_k\right)_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \left(\{c_k\} \cup C_k^{\mathcal{A}} \right) \\ m = \left(m_k\right)_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \left(M_k^{c'_k} \right) \end{array} \right\}.$$

i.e., we translate j's information (at the beginning of Stage 3) in the $\left[C^R, C^{\mathcal{A}}\right]$-game to that in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game via the function Σ_j.

We are now ready to replicate t_j with $t_j^{(c,s,t)}$.

$$t_j^{(c,s,t)} [q_j] \equiv t_j \left[\Sigma_j (q_j)\right], \forall q_j \in Q_j.$$

Similarly, we replicate b_j with $b_j^{(c,s,t)}$. When principal j observes $q_j = \left(c_j^{q_j}, s^{(c,s,t)} \left[\left(c_j^{q_j}, c_{-j}^{(c,s,t)} \right), \theta^{q_j} \right] \right) \in Q_j^*$, principal j's belief is induced by Bayes' rule, i.e.,

$$b_j^{(c,s,t)} (q_j) \left[\left\{ \left[c_{-j}^{(c,s,t)}, \ \theta' \right] \right\} \right] \equiv \begin{cases} \frac{p(\theta')}{p[\Upsilon(\theta)]} & \text{if } \theta' \in \Upsilon\left(\theta^{q_j}\right); \\ \\ 0 & \text{otherwise} \end{cases}, \forall \theta' \in \Theta,$$

where

$$\Upsilon\left(\theta^{q_j}\right) \equiv \left\{\widetilde{\theta} \in \Theta : q_j = \left(c_j^{q_j}, s^{(c,s,t)}\left[\left(c_j^{q_j}, c_{-j}^{(c,s,t)}\right), \widetilde{\theta}\right]\right)\right\}.$$

For each $k \in \mathcal{J}$ and each $y_k \in Y_k$, let $c_k^{y_k} \in C_k^{\mathcal{A}}$ denote the degenerate contract such that $c_k^{y_k}(m_k) = \{y_k\}$ for every $m_k \in M_k^{\mathcal{A}}$. When principal j observes $q_j \in Q_j \diagdown Q_j^*$, we have $\Sigma_j(q_j) = \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right)$, and define

$$b_j^{(c,s,t)}[q_j]\left[\left\{\left(\left(c_k^{y_k}\right)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right)\right\}\right]$$
$$\equiv b_j\left[\Sigma_j(q_j)\right]\left[\left\{\left(\left(c'_k\right)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right) : \begin{array}{l} \left(c'_k\right)_{k\in\mathcal{J}\diagdown\{j\}} \in C_{-j}^{\mathcal{A}} \text{ and} \\ \left(y_k\right)_{k\in\mathcal{J}\diagdown\{j\}} = \left(t_k\left[c'_k, \widetilde{m}^{q_j}\right]\right)_{k\in\mathcal{J}\diagdown\{j\}} \end{array}\right\}\right],$$
$$\forall\left[\left(y_k\right)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right] \in \left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta.$$

i.e., $b_j^{(c,s,t)}[q_j]$ mimics $b_j\left[\Sigma_j(q_j)\right]$ regarding induced belief on $\left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta$.

It is straightforward to see that (c, s, t) is replicated by $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$, and all the players inherit incentive compatibility from (c, s, t), i.e., $\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)$ is a $\left[C^R, C^{\mathcal{A}}\right]$-equilibrium and $z^{(c,s,t)} = z^{\left(c^{(c,s,t)}, s^{(c,s,t)}, t^{(c,s,t)}\right)}$.■

3.2 Proof of Proposition 2

Fix $\langle\mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle\mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public}\right\rangle$, and we aim to prove

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^R, C^{\mathcal{A}}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^R, C^{F^*}\right]}}.$$

Fix any $(c, s, t) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^R, C^{\mathcal{A}}\right]}$. We aim to replicate (c, s, t) with some $\left(c, \overline{s}, \overline{t}\right) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^R, C^{F^*}\right]}$ such that $z^{(c,s,t)} = z^{\left(c,\overline{s},\overline{t}\right)}$.

<u>Replicating $s \equiv (s_k)_{k\in\mathcal{J}}$ with $\overline{s} \equiv (\overline{s}_k)_{k\in\mathcal{J}}$:</u>

We replicate s in the $\left[C^R, C^{\mathcal{A}}\right]$-game with $\overline{s}$ in the $\left[C^R, C^{F^*}\right]$-game. Since s is defined on $\left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$ and $\overline{s}$ is defined on $\left(\{c_k\} \cup C_k^{F^*}\right)_{k\in\mathcal{J}}$, we need to define a function $\gamma : \left(\{c_k\} \cup C_k^{F^*}\right)_{k\in\mathcal{J}} \longrightarrow \left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$, such that, upon observing $c' \in \left(\{c_k\} \cup C_k^{F^*}\right)_{k\in\mathcal{J}}$ in the $\left[C^R, C^{F^*}\right]$-game, the agent regards it as $\gamma(c')$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game, and then, $\overline{s}(c')$ mimics $s[\gamma(c')]$. Thus, for each $j \in \mathcal{J}$, we define $\gamma_j : \{c_j\} \cup C_j^{F^*} \longrightarrow \{c_j\} \cup C_j^{\mathcal{A}}$ in three steps. First,

$$\gamma_j(c_j) = c_j.$$

For each $y_j \in Y_j$, let $c_j^{y_j}$ denote the degenerate contract such that $c_k^{y_k}(m_k) \equiv \{y_k\}$. Second for any $c_j^{y_j} \in C_j^{F^*} \diagdown \{c_j\}$, we define $\gamma_j\left(c_j^{y_j}\right)$ as $c_j^{y_j}$ in $C_j^{\mathcal{A}}$. Third, for any $c_j' \in C_j^F \diagdown \{c_j\}$, we have a pair of $\left[L_j \subset 2^{Y_j} \text{ and } H_j = \{[E_j, y_j] : y_j \in E_j\}\right]$ satisfying Definition 4 in Han and Xiong (2022). Fix any injective function $\phi_j^{c_j'} : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}$, i.e., given $c_j' \in C_j^F$, we identify a message $m_j' \in L_j \cup H_j$ in the $\left[C^R, C^{F^*}\right]$-game to the message $\phi_j^{c_j'}\left(m_j'\right) \in M_j^{\mathcal{A}}$ in the $\left[C^R, C^{\mathcal{A}}\right]$-game. With slight abuse of notation, let $\left(\phi_j^{c_j'}\right)^{-1} : \phi_j^{c_j'}\left[L_j \cup H_j\right] \longrightarrow L_j \cup H_j$ denote the inverse function of $\phi_j^{c_j'}$, i.e.,

$$\phi_j^{c_j'}\left[\left(\phi_j^{c_j'}\right)^{-1}(m_j)\right] = m_j,\ \forall m_j \in \phi_j^{c_j'}\left[L_j \cup H_j\right].$$

We thus define $\gamma_j\left(c_j'\right)$ for each $c_j' \in C_j^F \diagdown \{c_j\}$ as follows.

$$\gamma_j\left(c_j'\right)[m_j] = \begin{cases} c_j'\left(\left(\phi_j^{c_j'}\right)^{-1}[m_j]\right) & \text{if } m_j \in \phi_j^{c_j'}\left[L_j \cup H_j\right]; \\ E_j & \text{otherwise,} \end{cases} \quad \forall m_j \in M_j^{\mathcal{A}},$$

i.e., we first embed the message space $L_j \cup H_j$ into $M_j^{\mathcal{A}}$ by $\phi_j^{c_j'}$; second, we copy c_j' with $\gamma_j\left(c_j'\right)$ on the embedded message space; third, all of the other messages in $M_j^{\mathcal{A}}$ are mapped to E_j. Furthermore, denote $\gamma(c') \equiv \left(\gamma_k\left(c_k'\right)\right)_{k\in\mathcal{J}}$ and $\phi^{c'} :\equiv \left[\phi_j^{c_j'} : L_j \cup H_j \longrightarrow M_j^{\mathcal{A}}\right]_{k\in\mathcal{J}}$.

We are now ready to define $\overline{s} \equiv \left(\overline{s}_k\right)_{k\in\mathcal{J}}$. For each $j \in \mathcal{J}$, define

$$\overline{s}_j\left(c', \theta\right) \equiv \begin{cases} s_j\left(\gamma\left(c'\right), \theta\right) & \text{if } c_j' \in \{c_j\} \cup \left\{c_j^{y_j} : y_j \in Y_j\right\}; \\ \left(\phi_j^{c_j'}\right)^{-1}\left[s_j\left(\gamma\left(c'\right), \theta\right)\right] & \begin{array}{l}\text{if } c_j' \notin \{c_j\} \cup \left\{c_j^{y_j} : y_j \in Y_j\right\} \\ \text{and } s_j\left(\gamma\left(c'\right), \theta\right) \in \phi_j^{c_j'}\left[L_j\right];\end{array} \\ \left[E_j,\ y_j = t_j\left(\Gamma_j\left(\gamma\left(c'\right)\right), \Psi_j\left(s\left(\gamma\left(c'\right), \theta\right)\right)\right)\right] & \text{otherwise.} \end{cases}$$

When the agent observes $c' \in \left(\{c_k\} \cup C_k^{F^*}\right)_{k\in\mathcal{J}}$ in the $\left[C^R, C^{F^*}\right]$-game, the agent translates it to $\gamma(c') \in \left(\{c_k\} \cup C_k^{\mathcal{A}}\right)_{k\in\mathcal{J}}$ being offered in the $\left[C^R, C^{\mathcal{A}}\right]$-game. Then, $\overline{s}_j(c', \theta)$ replicates $s_j(\gamma(c'), \theta)$: if $c_j' \in \{c_j\} \cup \left\{c_j^{y_j} : y_j \in Y_j\right\}$, we have $\overline{s}_j(c', \theta) = s_j(\gamma(c'), \theta)$; if $c_j' \notin \{c_j\} \cup \left\{c_j^{y_j} : y_j \in Y_j\right\}$ and $s_j(\gamma(c'), \theta) \in \phi_j^{c_j'}[L_j]$, $\overline{s}_j(c', \theta)$ mimics $s_j(\gamma(c'), \theta)$ subject to re-labeling of message by $\left(\phi_j^{c_j'}\right)^{-1}$; otherwise, the message $s_j(\gamma(c'), \theta)$ pins down the subset E_j at Stage 2 in the $\left[C^R, C^{\mathcal{A}}\right]$-game, and at stage 3, j would take the action $t_j\left(\Gamma_j(\gamma(c')), \Psi_j(s(\gamma(c'), \theta))\right)$, and hence, $\overline{s}_j(c', \theta)$ mimics this by choosing

$$\left[E_j,\ y_j = t_j\left(\Gamma_j\left(\gamma\left(c'\right)\right), \Psi_j\left(s\left(\gamma\left(c'\right), \theta\right)\right)\right)\right]$$

in the menu-of-menu-with-full-recommendation contract $c_j' \in C_j^F$.

Replicating (b_j, t_j) with $(\bar{b}_j, \bar{t}_j)$:

Given $\langle \Gamma^{private}, \Psi^{public} \rangle$, each principal j observes only (c_j', m'). Thus, for notational simplicity, we write $t_j(c_j', m')$ and $b_j(c_j', m')$ for $t_j(\Gamma_j(c'), \Psi_j(m'))$ and $b_j(\Gamma_j(c'), \Psi_j(m'))$, respectively. Furthermore, j has a degenerate belief on $C_j \times M$, and hence, we describe only the marginal belief on $C_{-j} \times \Theta$.

Given $c_j' \in \{c_j\} \cup C_j^{F^*}$, let $M_j^{c_j'}$ denote the domain of c_j'. Consider

$$Q_j \equiv \left\{ (\widehat{c}_j, \widehat{m}) : \begin{array}{c} \exists \widehat{c} = \left(\{c_k\} \cup C_k^{F^*} \right)_{k \in \mathcal{J}} \\ m = (m_k)_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \left(M_k^{\widehat{c}_k} \right) \end{array} \right\},$$

$$Q_j^* \equiv \left\{ (\widehat{c}_j, s^* [(\widehat{c}_j, c_{-j}), \theta]) \in Q_j : \widehat{c}_j \in \left(\{c_j\} \cup C_j^{F^*} \right) \text{ and } \theta \in \Theta \right\},$$

i.e., in the $[C^R, C^{F^*}]$-game, Q_j is the set of all possible information that principal j may observe before j takes an action at Stage 3; Q_j^* is the subset of information by which j cannot confirm that the other players have deviated.

For each $q_j \in Q_j^*$, fix any $(c_j^{q_j}, \theta^{q_j})$ such that

$$q_j = \left(c_j^{q_j}, \bar{s} \left[\left(c_j^{q_j}, c_{-j} \right), \theta^{q_j} \right] \right).$$

For each $q_j = (\widehat{c}_j, \widehat{m}) \in Q_j \diagdown Q_j^*$, fix any $(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}) \in \left(\{c_j\} \cup C_j^{\mathcal{A}} \right) \times M^{\mathcal{A}}$ such that

$$\widehat{c}_j(\widehat{m}_j) = \widetilde{c}_j^{q_j} \left(\widetilde{m}_j^{q_j} \right).$$

We record this as $\Sigma_j : Q_j \longrightarrow Q_j^{\mathcal{A}}$ such that

$$\Sigma_j(q_j) \equiv \begin{cases} \left[\gamma_j(c^{q_j}), \ \bar{s} \left[\left(c_j^{q_j}, c_{-j} \right), \theta^{q_j} \right] \right] & \text{if } q_j \in Q_j^*; \\ \\ \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j} \right) & \text{if } q_j \in Q_j \diagdown Q_j^* \end{cases},$$

$$\text{where } Q_j^{\mathcal{A}} \equiv \left\{ (\Gamma_j(c'), \Psi_j(m)) : \begin{array}{c} c' = (c_k')_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \left(\{c_j\} \cup C_k^{\mathcal{A}} \right) \\ m = (m_k)_{k \in \mathcal{J}} \in \times_{k \in \mathcal{J}} \left(M_k^{c_k'} \right) \end{array} \right\}.$$

i.e., we translate j's information (at the beginning of Stage 3) in the $[C^R, C^{F^*}]$-game to that in the $[C^R, C^{\mathcal{A}}]$-game via the function Σ_j.

We are now ready to replicate t_j with $\bar{t}_j$.

$$\bar{t}_j[q_j] \equiv t_j[\Sigma_j(q_j)], \forall q_j \in Q_j.$$

Similarly, we replicate b_j with $\overline{b}_j$. When principal j observes $q_j = \left(c_j^{q_j}, \overline{s}\left[\left(c_j^{q_j}, c_{-j}\right), \theta^{q_j}\right]\right) \in Q_j^*$, principal j's belief is induced by Bayes' rule, i.e.,

$$\overline{b}_j\left(q_j\right)\left[\left\{\left[c_{-j},\ \theta'\right]\right\}\right] \equiv \begin{cases} \frac{p(\theta')}{p[\Upsilon(\theta)]} & \text{if } \theta' \in \Upsilon\left(\theta^{q_j}\right); \\ \\ 0 & \text{otherwise} \end{cases}, \forall \theta' \in \Theta,$$

where

$$\Upsilon\left(\theta^{q_j}\right) \equiv \left\{\widetilde{\theta} \in \Theta : q_j = \left(c_j^{q_j}, \overline{s}\left[\left(c_j^{q_j}, c_{-j}\right), \widetilde{\theta}\right]\right)\right\}.$$

For each $k \in \mathcal{J}$ and each $y_k \in Y_k$, let $c_k^{y_k} \in C_k^{F^*}$ denote the degenerate contract such that $c_k^{y_k}\left(m_k\right) \equiv \{y_k\}$. When principal j observes $q_j \in Q_j \diagdown Q_j^*$, we have $\Sigma_j\left(q_j\right) = \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right)$, and define

$$\overline{b}_j\left[q_j\right]\left[\left\{\left(\left(c_k^{y_k}\right)_{k \in \mathcal{J}\diagdown\{j\}}, \theta\right)\right\}\right]$$
$$\equiv b_j\left[\Sigma_j\left(q_j\right)\right]\left[\left\{\left(\left(c_k'\right)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right) : \begin{array}{l}\left(c_k'\right)_{k\in\mathcal{J}\diagdown\{j\}} \in C_{-j}^{\mathcal{A}} \text{ and} \\ \left(y_k\right)_{k\in\mathcal{J}\diagdown\{j\}} = \left(t_k\left[c_k', \widetilde{m}^{q_j}\right]\right)_{k\in\mathcal{J}\diagdown\{j\}}\end{array}\right\}\right],$$
$$\forall \left[\left(y_k\right)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right] \in \left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta.$$

i.e., $\overline{b}_j\left[q_j\right]$ mimics $b_j\left[\Sigma_j\left(q_j\right)\right]$ regarding induced belief on $\left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta$.

It is straightforward to see that (c, s, t) is replicated by $\left(c, \overline{s}, \overline{t}\right)$, and all the players inherit incentive compatibility from (c, s, t), i.e., $\left(c, \overline{s}, \overline{t}\right)$ is a $\left[C^R, C^{F^*}\right]$-equilibrium and $z^{(c,s,t)} = z^{\left(c,\overline{s},\overline{t}\right)}$.■

3.3 Proof of Proposition 3

Fix $\langle\mathcal{A},\ \Gamma,\ \Psi\rangle = \left\langle\mathcal{A}^{non-delegated}, \Gamma^{private}, \Psi^{public}\right\rangle$, and we aim to prove

$$Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{F^*}\right]}} \subset Z^{\mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}}.$$

Fix any $(c, s, t) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{F^*}\right]}$. We aim to replicate (c, s, t) with some $(c, s^*, t^*) \in \mathcal{E}^{\langle\mathcal{A},\ \Gamma,\ \Psi\rangle\text{-}\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]}$ such that $z^{(c,s,t)} = z^{(c,s^*,t^*)}$.

<u>Replicating $s \equiv \left(s_k\right)_{k\in\mathcal{J}}$ with $s^* \equiv \left(s_k^*\right)_{k\in\mathcal{J}}$:</u>

Since $C^{\mathcal{A}} \sqsupset^{**} C^{F^*}$, there exists a function $\psi_j : C_j^{\mathcal{A}} \longrightarrow C_j^{F^*}$ for each $j \in \mathcal{J}$ such that

$$c_j' \geq \psi_j\left(c_j'\right), \forall c_j' \in C_j^{\mathcal{A}}.$$

Thus, for each $c'_j \in C_j^{\mathcal{A}}$, there exists a surjective $\iota_j^{c'_j} : M_j^{\mathcal{A}} \longrightarrow M_j^{F^*}$ such that

$$c'_j(m_j) = \psi_j(c'_j)\left(\iota_j^{c'_j}(m_j)\right), \forall m_j \in M_j^{\mathcal{A}}.$$

Let $\left(\iota_j^{c'_j}\right)^{-1} : M_j^{F^*} \longrightarrow M_j^{\mathcal{A}}$ denote any injective function such that

$$c'_j\left(\left(\iota_j^{c'_j}\right)^{-1}(m_j)\right) = \psi_j(c'_j)(m_j), \forall m_j \in M_j^{F^*}.$$

When the agent observes c_j in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, he interprets it as c_j in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game. Also, when the agent observes $c'_j \in C_j^{\mathcal{A}} \diagdown \{c_j\}$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, he interprets it as $\psi_j(c'_j) \in C_j^{F^*}$ in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game. To record this interpretation, define $\gamma_j : C_j^{\mathcal{A}} \longrightarrow \left(\{c_j\} \cup C_j^{F^*}\right)$ for each $j \in \mathcal{J}$ as

$$\gamma_j(c'_j) \equiv \begin{cases} c_j, & \text{if } c'_j = c_j; \\ \psi_j(c'_j), & \text{if } c'_j \in C_j^{\mathcal{A}} \diagdown \{c_j\}, \end{cases}$$

and denote $\gamma(c') \equiv \left([\gamma_k(c'_k)]_{k\in\mathcal{J}}\right) \in \left(\{c_k\} \cup C_k^{F^*}\right)_{k\in\mathcal{J}}$ for all $c' \in C^{\mathcal{A}}$.

For the agent's strategy in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, we replicate $s \equiv (s_k)_{k\in\mathcal{J}}$ with $s^* \equiv (s_k^*)_{k\in\mathcal{J}}$ defined as follows. For each $j \in \mathcal{J}$ and all $(c', \theta) \in C^{\mathcal{A}} \times \Theta$,

$$s_j^*(c', \theta) = \begin{cases} s_j[\gamma(c'),\ \theta], & \text{if } c'_j = c_j; \\ \left(\iota_j^{\gamma_j(c'_j)}\right)^{-1}(s_j[\gamma(c'),\ \theta]), & \text{if } c'_j \in C_j^{\mathcal{A}} \diagdown \{c_j\}. \end{cases}$$

I.e., when principals offer c' in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, the agent regards it as $\gamma(c')$ in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game. Given $\gamma(c')$ in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game, the agent sends $s_j[\gamma(c'),\ \theta]$ to j at Stage 2. If $c'_j = c_j$, the agent indeed sends $s_j[\gamma(c'),\ \theta]$. If $c'_j \in C_j^{\mathcal{A}} \diagdown \{c_j\}$, c'_j corresponds to $\gamma_j(c'_j)$ in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game, and the equilibrium message $s_j[\gamma(c'),\ \theta]$ corresponds to the message $\left(\iota_j^{\gamma_j(c'_j)}\right)^{-1}(s_j[\gamma(c'),\ \theta])$ in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game.

Extending $\left(\widehat{t}, \widehat{b}\right)$ to (t^*, b^*):

Given $\left\langle \Gamma^{private}, \Psi^{public} \right\rangle$, each principal j observes only $\left(c'_j, m'\right)$. Thus, for notational simplicity, we write $t_j\left(c'_j, m'\right)$ and $b_j\left(c'_j, m'\right)$ for $t_j(\Gamma_j(c'), \Psi_j(m'))$ and $b_j(\Gamma_j(c'), \Psi_j(m'))$, respectively. Furthermore, j has a degenerate belief on $C_j \times M$, and hence, we describe only the marginal belief on $C_{-j} \times \Theta$. Consider

$$Q_j \equiv \left\{(\widehat{c}_j, \widehat{m}) : \widehat{c}_j \in C_j^{\mathcal{A}} \text{ and } \widehat{m} \in M^{\mathcal{A}}\right\},$$

$$Q_j^* \equiv \left\{ \left(\widehat{c}_j, s^*\left[\left(\widehat{c}_j, c_{-j}\right), \theta\right]\right) \in Q_j : \widehat{c}_j \in C_j^{\mathcal{A}} \text{ and } \theta \in \Theta \right\},$$

i.e., in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game, Q_j is the set of all possible information that principal j may observe before j takes an action at Stage 3; Q_j^* is the subset of information by which j cannot confirm that the other players have deviated.

For each $q_j \in Q_j^*$, fix any $\left(c_j^{q_j}, \theta^{q_j}\right)$ such that

$$q_j = \left(c_j^{q_j}, s^*\left[\left(c_j^{q_j}, c_{-j}\right), \theta^{q_j}\right]\right).$$

For each $q_j = (\widehat{c}_j, \widehat{m}) \in Q_j \diagdown Q_j^*$, fix any $\left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right) \in \left(\{c_j\} \cup C_j^{F^*}\right) \times \left(M_j^{c_j} \cup M_j^{F^*}\right)$ such that

$$\widehat{c}_j\left(\widehat{m}_j\right) = \widetilde{c}_j^{q_j}\left(\widetilde{m}_j^{q_j}\right).$$

We record this as $\Sigma_j : Q_j \longrightarrow Q_j^{F^*}$ such that

$$\Sigma_j\left(q_j\right) \equiv \begin{cases} \left[\gamma_j\left(c^{q_j}\right),\ s^*\left[\left(c_j^{q_j}, c_{-j}\right), \theta^{q_j}\right]\right] & \text{if } q_j \in Q_j^*; \\ \\ \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right) & \text{if } q_j \in Q_j \diagdown Q_j^* \end{cases},$$

$$\text{where } Q_j^{F^*} \equiv \left\{ \left(\Gamma_j\left(c'\right), \Psi_j\left(m\right)\right) : \begin{array}{c} c' = \left(c'_k\right)_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}}\left(\{c_j\} \cup C_k^{F^*}\right) \\ m = \left(m_k\right)_{k\in\mathcal{J}} \in \times_{k\in\mathcal{J}}\left(M_k^{c'_k}\right) \end{array} \right\}.$$

i.e., we translate j's information (at the beginning of Stage 3) in the $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-game to that in the $\left[C^{\mathcal{A}}, C^{F^*}\right]$-game via the function Σ_j.

We are now ready to replicate t_j with t_j^*.

$$t_j^*\left[q_j\right] \equiv t_j\left[\Sigma_j\left(q_j\right)\right], \forall q_j \in Q_j.$$

Similarly, we replicate b_j with b_j^*. When principal j observes $q_j = \left(c_j^{q_j}, s^*\left[\left(c_j^{q_j}, c_{-j}\right), \theta^{q_j}\right]\right) \in Q_j^*$, principal j's belief is induced by Bayes' rule, i.e.,

$$b_j^*\left(q_j\right)\left[\left\{\left[c_{-j},\ \theta'\right]\right\}\right] \equiv \begin{cases} \frac{p(\theta')}{p[\Upsilon(\theta)]} & \text{if } \theta' \in \Upsilon\left(\theta^{q_j}\right); \\ \\ 0 & \text{otherwise} \end{cases}, \forall \theta' \in \Theta,$$

where

$$\Upsilon\left(\theta^{q_j}\right) \equiv \left\{ \widetilde{\theta} \in \Theta : q_j = \left(c_j^{q_j}, s^*\left[\left(c_j^{q_j}, c_{-j}\right), \widetilde{\theta}\right]\right) \right\}.$$

For each $k \in \mathcal{J}$ and each $y_k \in Y_k$, let $c_k^{y_k} \in C_k^{\mathcal{A}}$ denote the degenerate contract such that $c_k^{y_k}(m_k) = \{y_k\}$ for every $m_k \in M_k^{\mathcal{A}}$. When principal j observes $q_j \in Q_j \diagdown Q_j^*$, we have $\Sigma_j(q_j) = \left(\widetilde{c}_j^{q_j}, \widetilde{m}^{q_j}\right)$, and define

$$b_j^*[q_j]\left[\left\{\left((c_k^{y_k})_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right)\right\}\right]$$
$$\equiv b_j[\Sigma_j(q_j)]\left[\left\{\left((c'_k)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right) : \begin{array}{l} (c'_k)_{k\in\mathcal{J}\diagdown\{j\}} \in C_{-j}^{\mathcal{A}} \text{ and} \\ (y_k)_{k\in\mathcal{J}\diagdown\{j\}} = \left(t_k\left[c'_k, \widetilde{m}^{q_j}\right]\right)_{k\in\mathcal{J}\diagdown\{j\}} \end{array}\right\}\right],$$
$$\forall \left[(y_k)_{k\in\mathcal{J}\diagdown\{j\}}, \theta\right] \in \left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta.$$

i.e., $b_j^*[q_j]$ mimics $b_j[\Sigma_j(q_j)]$ regarding induced belief on $\left[\times_{k\in\mathcal{J}\diagdown\{j\}} Y_k\right] \times \Theta$.

It is straightforward to see that (c, s, t) is replicated by (c, s^*, t^*), and all the players inherit incentive compatibility from (c, s, t), i.e., (c, s^*, t^*) is a $\left[C^{\mathcal{A}}, C^{\mathcal{A}}\right]$-equilibrium and $z^{(c,s,t)} = z^{(c,s^*,t^*)}$.■

References

HAN, S., AND S. XIONG (2022): "Common Agency with Non-Delegation or Imperfect Commitment," *mimeo*.