Robust Competitive Auctions

Seungjin Han[*]

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Abstract

A competitive distribution of auctions (Peters 1997) is robust to the possibility of a seller's deviation to any arbitrary mechanisms, let alone direct mechanisms because the sufficient condition for the robustness is embedded in its notion of equilibrium.

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1 Introduction

Peters (1997) studies a decentralized auction market with many sellers and buyers, all of whom differ in terms of their valuation for the item being traded. To highlight frictions in the market, he focuses on an *incentive consistent continuation equilibrium* in which (i) buyers use the same strategy (symmetry) and (ii) they choose non-deviating sellers with equal probability if their auctions are the same (non-discrimination among non-deviating sellers). His result shows that when every seller offers a second price auction with reserve price equal to his cost, a seller cannot improve his profits by offering any alternative direct mechanism instead of his second price auction as the number of sellers increases to the infinity. However, it is not yet known whether a seller can gain by deviating to an arbitrary mechanism other than a direct mechanism.[1]

Fix the distribution of second price auctions offered by all sellers except a deviating seller. In any incentive consistent continuation equilibrium upon a seller's deviation to any arbitrary (indirect) mechanism, one can always extract a (Bayesian) incentive compatible direct mechanism from the deviating seller's mechanism by using the buyers' strategies of communicating with the deviating seller.[2] If the deviating seller directly deviates to that incentive compatible direct mechanism, one can derive a payoff-equivalent incentive consistent continuation equilibrium where the buyers maintain the original probabilities of selecting the deviating seller. Therefore, the deviating seller cannot gain by deviating to any arbitrary mechanism if he cannot gain in every incentive consistent continuation equilibrium upon offering every possible incentive compatible direct mechanism.

This sufficient condition is embedded in the notion of a competitive distribution of auctions in Peters (1997), so a competitive distribution of auctions is robust to the possibility that sellers may deviate to any arbitrary mechanisms, not just direct mechanisms. The key for the robustness is that for any given incentive compatible direct mechanism that a seller may deviate to, there may be multiple incentive consistent continuation equilibria, which differ in terms of the buyers' strategies of selecting him and hence we need to consider all possible selection strategies that make the deviator's direct mechanism incentive compatible in order to check if a seller has an incentive to deviate to any arbitrary mechanism.

2 A Competitive Distribution of Auctions

In Peters (1997), J sellers face κJ buyers: $\mathcal{J} = \{1, \ldots, J\}$ and $\mathcal{I} = \{J+1, \ldots, (\kappa+1)J\}$. Each seller has one unit of an indivisible good to sell. Each buyer needs one unit of the good

[1] The standard Revelation Principle is not readily applicable to a market with many sellers (principals) because buyers (agents) are informed not only about their valuation but also about selling mechanisms offered by competing sellers. Therefore, the message space only over buyers' possible valuations in a direct mechanism may not be large enough to encompass all the information that buyers have in the market. Epstein and Peters (1999) propose a mechanism with the universal language that allows buyers to describe competing seller's selling mechanism and establish a revelation principle with this type of universal mechanisms. However, it is not easy to apply universal mechanisms.

[2] Regardless of the selling mechanism offered by the deviating seller, we can conveniently fix buyers' truthful type reporting to the other sellers upon selecting them because their selling mechanisms, second price auctions, are dominant strategy incentive compatible.

and the distribution of the buyer's valuation is denoted by F with its support $[0,1]$. Each seller has a cost associated with selling his good. The distribution of costs in the population of sellers is denoted by G with its support contained in $[0,1]$. In seller j's anonymous direct mechanism, the message space for each buyer is $\bar{X} = [0,1] \cup \{x^\circ\}$, where x° denotes that the buyer does not participate in the mechanism. Suppose that x is a buyer's message and that $\mathbf{x}$ is the profile of the other buyers' messages. Let $p_j(x, \mathbf{x})$ denote the price that a buyer pays to seller j and $q_j(x, \mathbf{x})$ the probability with which a buyer acquires the object. Then, seller j's direct mechanism is denoted by $\mu_j = \{p_j, q_j\}$.

Suppose that π denotes a buyer's incentive consistent selection strategy.[3] In particular, $\pi(x, \mu_j, \mu_{-j}) \in [0,1]$ specifies the probability with which a buyer with valuation x selects seller j who offers μ_j given the other sellers' mechanisms μ_{-j}. Given $\mu = (\mu_j, \mu_{-j})$ and π, let $z_j(\mu, \pi)(x)$ denote the probability that a buyer's valuation is less than x or selects a seller other than j. Then, $z_j(\mu, \pi)$ can be derived as follows[4]: For all $x \in [0,1]$,

$$z_j(\mu, \pi)(x) = 1 - \int_x^1 \pi(s, \mu_j, \mu_{-j}) f(s) ds. \tag{1}$$

One can use $z_j(\mu, \pi)$ to derive the reduced-form probability $Q_j(x, \mu, \pi)$ with which a buyer with valuation x expects to acquire the object as follows:

$$Q_j(x, \mu_j, \pi) = \int_0^1 \cdots \int_0^1 q_j(x, s_{J+2}, \ldots s_{(\kappa+1)J}) dz_j(\mu, \pi)(s_{J+2}) \ldots dz_j(\mu, \pi)(s_{(\kappa+1)J}), \tag{2}$$

Similarly we can derive $P_j(x, \mu_j, \pi)$, the reduced-form price that she expects to pay upon selecting seller j. Therefore, $z_j(\mu, \pi)$ determines the reduced-form mechanism, $Q_j(x, \mu_j, \pi)$ and $P_j(x, \mu_j, \pi)$.

Peters (1997) considers the finite approximation of the limit game with the infinite number of traders given the fixed ratio κ of buyers to sellers.[5] A cutoff valuation for seller j is the infimum of the set of valuations for which buyers choose seller j with positive probability. Let H denote a distribution of cutoff valuations for the limit game.

For a finite approximation, consider the market where $J-1$ sellers hold *second-price auctions* $\bar{\mu}_{-J} = \{\bar{\mu}_1, \ldots, \bar{\mu}_{J-1}\}$ with the distribution of the cutoff valuations $\bar{H}_J$ that converges almost everywhere to H. Let $\pi'_J(x, \mu'_J)$ be the selection probability with which a buyer with valuation x selects deviating seller J when his direct mechanism is μ'_J given $\bar{\mu}_{-J}$. According to Peters (1997), $\pi'_J(\cdot, \mu'_J)$ alone completely determines the incentive consistent strategies of selecting non-deviating sellers.[6]

[3]An incentive consistent strategy strategy implies a strategy that buyers use in an incentive consistent continuation equilibrium.

[4]If not confused, some of the notations are slightly modified from those in Peters (1997).

[5]This is because, as Peters and Severinov (1997) suggest, competing auction games often do not admit a pure-strategy equilibrium with the finite number of sellers. Burguet and Sákovics (1999) show the existence of mixed-strategy equilibrium in the two-seller case. Virág (2010) extends their result to any finite number of homogenous sellers and shows that when sellers' costs are equal to zero, equilibrium reserve prices converge to zero.

[6]Unless specified, we will use π'_J when we refer to the buyer's incentive consistent selection strategy.

Then, one can calculate the payoff, $\bar{v}_1(x, \mu'_J, J)$, to a buyer with valuation x by selecting the non-deviating seller offering the lowest reserve price:

$$\bar{v}_1(x, \mu'_J, J) = \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \pi'_J(x, \mu'_J)}{n_J(s, H)} f(s)ds\right]^{\kappa J - 1} d\nu, \tag{3}$$

where $n_J(x, H) = \max\{j : \bar{y}_j \leq x\}$ and $\frac{1-\pi'_J(x,\mu'_J)}{n_J(s,H)}$ is the selection probability with which a buyer with valuation x chooses the non-deviating seller with the lowest reserve price.[7] Peters (1997) shows that every non-deviating seller will give the same expected payoff as the non-deviating seller who offers the lowest reserve price when the matching process is incentive consistent.

Then, for any incentive consistent selection strategy π'_J, the deviating seller's payoff associated with offering an incentive compatible direct mechanism is

$$\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = w + \kappa J \int_0^1 \left[(x - w)Q_J(x, \mu'_J, \pi'_J) - \bar{v}_1(x, \mu'_J, J)\right] \pi'_J(x, \mu'_J) f(x)dx. \tag{4}$$

Finally, let $\hat{\Phi}'_J(w, \mu'(y), H)$ denote the payoff to the deviating seller with cost w if he offers a second-price auction $\mu'(y)$ that induces a cutoff valuation y. Let $\Pi'_J(\mu'_J, H)$ be the set of all incentive consistent selection strategies π'_J that lead to an incentive consistent continuation equilibrium given (μ'_J, H). Let $\mathcal{M}^B_J(H)$ be the set of all (Bayesian) incentive compatible direct mechanisms available for each seller's deviation given H. The definition of a competitive distribution of second-price auctions in Peters (1997) can be provided as follows:

Definition 1 *A competitive distribution of second-price auctions is a distribution of cutoff valuations H and a cutoff rule $y : [0, 1] \to [0, 1]$ such that for almost all w,*

1. *for all $\mu'_J \in \mathcal{M}^B_J(H)$ and all $\pi'_J \in \Pi'_J(\mu'_J, H)$*

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \tag{5}$$

2. *and $H(y(w)) = G(w)$.*

Theorem 5 in Peters (1997) shows that there is a competitive distribution of second-price auctions in which each seller offers a second-price auction with reserve price equal to his cost.

3 Robustness

A seller may want to offer an arbitrary mechanism, not necessarily a direct mechanism. Suppose that, given other sellers' auction offers, seller J deviates to an anonymous mechanism $\gamma_J = \{p^\gamma_J, q^\gamma_J\}$, where $p^\gamma_J : M^{\kappa J} \to \mathbb{R}$ specifies a buyer's payment and $p^\gamma_J : M^{\kappa J} \to [0, 1]$

[7] See Lemma 2 in Peters (1997) for details.

specifies the probability that a buyer acquires the item, with M being the set of messages that a buyer can send to the seller.[8] Let $m^\circ \in M$ be the message for "no participation."

Let $c'_J : X \to \Delta(M)$ be the strategy that every buyer uses for communicating with seller J upon selecting him. Given c'_J, one can induce a direct mechanism $\beta^c(\gamma_J) = \{p^c_J, q^c_J\}$ as follows: For all $N \leq \kappa J$,

$$p^c_J(\mathbf{x}_N, \mathbf{x}^\circ_{-N}) = \int_M \cdots \int_M p^\gamma_J(\mathbf{m}_N, \mathbf{m}^\circ_{-N}) dc'_J(x_1) \times \cdots \times dc'_J(x_N), \tag{6}$$

$$q^c_J(\mathbf{x}_N, \mathbf{x}^\circ_{-N}) = \int_M \cdots \int_M q^\gamma_J(\mathbf{m}_N, \mathbf{m}^\circ_{-N}) dc'_J(x_1) \times \cdots \times dc'_J(x_N), \tag{7}$$

where $\mathbf{x}_N = (x_1, \ldots, x_N)$ is the profile of N participating buyers' valuations, $\mathbf{m}_N = (m_1, \ldots m_N)$ is the profile of messages sent by participating buyers, and $\mathbf{x}^\circ_{-N}$ and $\mathbf{m}^\circ_{-N}$ are the profiles of the messages that are equivalent to "no participation."

Let $\acute{\pi}^\gamma_J(x, \gamma_J) \in [0, 1]$ denote the probability with which a buyer with valuation x selects J. One can derive $z_J(x, \gamma_J, \acute{\pi}^\gamma_J)$ similar to (1) and the reduced-form mechanism $\{Q^c_J(x, \gamma_J, \acute{\pi}^\gamma_J), P^c_J(x, \gamma_J, \acute{\pi}^\gamma_J)\}$ similar to (2). Because of Lemma 2 in Peters (1997), the buyer's incentive consistent strategy of selecting non-deviating sellers is fully determined by the strategy $\acute{\pi}^\gamma_J$ of selecting deviating seller J. Therefore, we will use $(c'_J, \acute{\pi}^\gamma_J)$ when we refer to the buyer's incentive consistent strategy. A buyer's payoff associated with selecting a seller with the lowest reserve price is derived as follows:

$$\bar{v}_1(x, \gamma_J, J) = \int_{y_1}^x \left[1 - \int_\nu^1 \frac{1 - \acute{\pi}^\gamma_J(x, \gamma_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu,$$

where $\frac{1 - \acute{\pi}^\gamma_J(x, \gamma_J)}{n_J(s, H)}$ is the probability with which a buyer with valuation x chooses the non-deviating seller with the lowest reserve price.[9] Given $\bar{v}_1(x, \gamma_J, J)$, seller J's payoff associated with offering an arbitrary mechanism γ_J is

$$\Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J) = w + \kappa J \int_0^1 \left[(x - w) Q^c_J(x, \gamma_J, \acute{\pi}^\gamma_J) - \bar{v}_1(x, \gamma_J, J)\right] \acute{\pi}^\gamma_J(x, \gamma_J) f(x) dx. \tag{8}$$

Following the notion of robustness proposed by Epstein and Peters (1999), we define the robustness of a competitive distribution of second-price auctions. Let Γ be a set of arbitrary mechanisms that a seller may consider for his deviation.

Definition 2 *A competitive distribution of second-price auctions is said to be robust to a set of mechanisms Γ if, for all $(\acute{\pi}^\gamma_J, c'_J)$ that characterizes an incentive consistent continuation equilibrium upon a seller's deviation to any mechanism in Γ, the following condition is satisfied: for every $\gamma_J \in \Gamma$,*

$$\lim_{J \to \infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J \to \infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J).$$

[8] As any direct mechanism is assumed to be anonymous in Peters (1997), any arbitrary mechanism is also assumed to be anonymous.

[9] See Lemma 2 in Peters (1997).

3.1 Robust Competitive Auctions

Lemma 1 is instrumental in establishing the robustness of competitive second-price auctions.

Lemma 1 *For any* $(c'_J, \acute{\pi}^{\gamma}_J)$ *that characterizes an incentive consistent continuation equilibrium upon seller* J*'s deviation to an arbitrary mechanism* $\gamma_J \in \Gamma$*, there exist an incentive compatible direct mechanism* μ'_J *and an incentive consistent selection strategy* π'_J *such that*

$$\hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J). \tag{9}$$

Proof. We assume that every buyer always reports her true type to non-deviating sellers upon selecting them because their selling mechanisms, second-price auctions, are dominant strategy incentive compatible. By using c'_J in $(c'_J, \acute{\pi}^{\gamma}_J)$, one can construct a direct mechanism $\beta^c(\gamma_J) = \{p^c_J, q^c_J\}$ according to (6) and (7). Suppose that seller J directly offers a direct mechanism

$$\mu'_J = \beta^c(\gamma_J)$$

instead of $\gamma_J \in \Gamma$. Assume that a buyer selects seller J according to

$$\pi'_J(x, \mu'_J) = \acute{\pi}^{\gamma}_J(x, \gamma_J) \tag{10}$$

for all x. After seller J's deviation to μ'_J, a buyer selects each non-deviating seller with the same probability that she would have selected him if seller J deviated to γ_J.

For the deviating seller J, notice that (10) means that the probability distribution z_J induced by π'_J given $\mu'_J = \beta^c(\gamma_J)$ (its derivation similar to (1)) is the same as z_J induced $\acute{\pi}^{\gamma}_J$. Because the reduced-form mechanisms are based on z_J, it implies that the reduced-form mechanism from γ_J is the same as the reduced mechanism from $\mu'_J = \beta^c(\gamma_J)$ given the construction of $\beta^c(\gamma_J) = \{p^c_J, q^c_J\}$ according to (6) and (7):

$$\begin{aligned} Q_J(x, \mu'_J, \pi'_J) &= Q^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J), \\ P_J(x, \mu'_J, \pi'_J) &= P^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J). \end{aligned}$$

Because the reduced-form mechanism is identical, the payoff to a buyer with valuation x by selecting seller J is preserved by truthful valuation reporting;

$$Q_J(x, \mu'_J, \pi'_J)x - P_J(x, \mu'_J, \pi'_J) = Q^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J)x - P^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J). \tag{11}$$

If a buyer with valuation x reports x' to seller J who offered μ'_J, her payoff is

$$Q_J(x', \mu'_J, \pi'_J)x - P_J(x', \mu'_J, \pi'_J) = Q^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J), \tag{12}$$

where the right-hand side is the buyer's payoff when she draws from the probability distribution $c'_J(x')$ her message that she sends to seller J who offers γ_J.

Given γ_J, suppose that a buyer with valuation x chooses a message for seller J from the probability distribution $c'_J(x')$ instead of $c'_J(x)$. Then, her payoff is $Q^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)$. Because $c'_J(x)$ is optimal for a buyer with valuation x, we have

$$Q^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J)x - P^c_J(x, \gamma_J, \acute{\pi}^{\gamma}_J) \geq Q^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J)x - P^c_J(x', \gamma_J, \acute{\pi}^{\gamma}_J). \tag{13}$$

(11) - (13) yields, for all $x, x' \in [0,1]$,

$$Q_J(x, \mu'_J, \pi'_J)x - P_J(x, \mu'_J, \pi'_J) \geq Q_J(x', \mu'_J, \pi'_J)x - P_J(x', \mu'_J, \pi'_J)$$

so that $\mu'_J = \beta^c(\gamma_J)$ is incentive compatible given π'_J. Because all sellers' mechanisms are incentive compatible given the buyer's same selection behavior and the buyer's payoff upon selecting every seller is also preserved, the incentive consistency of $\acute{\pi}^\gamma_J$ is preserved by π'_J.

Because the selection probability that a buyer would choose any seller is preserved, a buyer's payoff associated with selecting a seller with the lowest reserve price is preserved;

$$\begin{aligned} \bar{v}_1(x, \mu'_J, J) &= \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \pi'_J(x, \mu'_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu \\ &= \int_{y_1}^{x} \left[1 - \int_{\nu}^{1} \frac{1 - \acute{\pi}^\gamma_J(x, \gamma_J)}{n_J(s, H)} f(s) ds\right]^{\kappa J - 1} d\nu \\ &= \bar{v}_1(x, \gamma_J, J). \end{aligned} \tag{14}$$

(14) finally induces

$$\begin{gathered} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) = \\ w + \kappa J \int_0^1 \left[(x - w) Q_J(x, \mu'_J, \pi'_J) - \bar{v}_1(x, \mu'_J, J)\right] \pi'_J(x, \mu'_J) f(x) dx = \\ w + \kappa J \int_0^1 \left[(x - w) Q^c_j(x, \gamma_J, \acute{\pi}^\gamma_J) - \bar{v}_1(x, \gamma_J, J)\right] \acute{\pi}^\gamma_J(x, \gamma_J) f(x) dx = \\ \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J). \end{gathered} \tag{15}$$

Therefore, the deviating seller J's payoff is preserved. ■

We now establish the robustness of a competitive distribution of second-price auctions.

Proposition 1 *A competitive distribution of second-price auctions is robust to any seller's deviation to any mechanism in any* Γ.

Proof. For any $\gamma_J \in \Gamma$, let $\mu'_J = \beta^c(\gamma_J)$ be an incentive compatible direct mechanism that is induced by an incentive consistent strategy $(c'_J, \acute{\pi}^\gamma_J)$. Let $\Pi'_J(\mu'_J, H)$ be the set of all incentive consistent selection strategies π'_J given (μ'_J, H). (9) in Lemma 1 implies that, for any $\gamma_J \in \Gamma$,

$$\sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J). \tag{16}$$

Because μ'_J in (16) is incentive compatible (i.e., $\mu'_J \in \mathcal{M}^B_J(H)$), we have that

$$\sup_{\mu'_J \in \mathcal{M}^B_J(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J). \tag{17}$$

(16) and (17) yield that for every $\gamma_J \in \Gamma$,

$$\lim_{J \to \infty} \sup_{\mu'_J \in \mathcal{M}^B_J(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \geq \lim_{J \to \infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^\gamma_J). \tag{18}$$

The first part of the definition of a competitive distribution of auctions (i.e., Part 1 of Definition 1) implies that for every $\gamma_J \in \Gamma$,

$$\lim_{J\to\infty} \Phi'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \sup_{\mu'_J \in \mathcal{M}^B_J(H)} \sup_{\pi'_J \in \Pi'_J(\mu'_J, H)} \Phi_J(w, \mu'_J, H, \pi'_J). \tag{19}$$

Finally, from (18) and (19), we can conclude that for every $\gamma_J \in \Gamma$ and every incentive consistent strategy $(c'_J, \acute{\pi}^{\gamma}_J)$

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \Phi_J(w, \gamma_J, H, c'_J, \acute{\pi}^{\gamma}_J),$$

which satisfies the robustness in Definition 2. Therefore, a competitive distribution of auctions is robust. ■

After seller J deviates to an arbitrary mechanism γ_J, any incentive consistent continuation equilibrium $(c'_J, \acute{\pi}^{\gamma}_J)$ induces a (Bayesian) incentive compatible direct mechanism $\beta^c(\gamma_J)$.[10] Suppose that seller J directly deviates to the direct mechanism $\mu'_J = \beta^c(\gamma_J)$ instead of γ_J. For any given strategy $\acute{\pi}^{\gamma}_J$ of selecting J in the original incentive consistent continuation equilibrium upon J's deviation to γ_J, Lemma 1 shows that there exists the corresponding payoff-equivalent incentive consistent continuation equilibrium upon J's deviation to μ'_J where a buyer's selection strategy π'_J induces the same probability of selecting J as that in the original continuation equilibrium and makes μ'_J incentive compatible.

Therefore, seller J cannot gain by deviating to any arbitrary mechanism if he cannot gain in *every* incentive consistent continuation equilibrium upon offering every possible *incentive compatible direct mechanism*. The key for the robustness is that for any given incentive compatible direct mechanism μ_J that seller J may deviate to, there can be multiple incentive consistent continuation equilibria, which differ in the buyer's strategy of selecting him and hence we need to consider all possible incentive consistent selection strategies that make μ_J incentive compatible. This is incorporated in the definition of a competitive distribution of second-price auctions in Peters (1997).[11]

References

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[10] Bayesian incentive compatibility of $\beta^c(\gamma_J)$ is tied to $\acute{\pi}^{\gamma}_J$ because Bayesian incentive compatibility is based on a buyer's (interim) expected payoff that is derived by using the other buyers' selection strategies that specify probabilities with which they select seller J.

[11] The tractable feature of non-deviating sellers' second price auctions in the analysis of the equilibrium robustness is that we can always conveniently fix the buyer's communication with non-deviating sellers as truthful valuation report regardless of the deviating seller's mechanism due to the dominant strategy incentive compatibility of second price auctions. This approach can be extended to apply the sufficient condition for the equilibrium robustness if we were to consider other competing mechanism games where a principal is allowed to offer only a mechanism from a subset of dominant strategy incentive compatible (DIC) direct mechanisms (e.g., competing excludable public good provision). See the working paper version (Han 2014) for how to generally establish the sufficient condition for the equilibrium robustness in a competing DIC direct mechanism game.

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