Quasi Ex-Post Equilibrium in Competing Mechanisms

Seungjin Han*

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Abstract

This paper studies competing mechanism games with no restrictions on the complexity of mechanisms where principals can announce mechanisms and agents select and communicate with at most one principal. It proposes the solution concept of robust quasi ex-post equilibrium in which agents' strategies of communicating with a non-deviating principal is ex-post optimal. Two simple revelation principles are established. The Strong Revelation Principle allows us to check if an equilibrium allocation in a specific competition model is a robust quasi ex-post equilibrium allocation. The Weak Revelation Principle leads to the characterization of the set of robust quasi ex-post equilibrium allocations in terms of model primitives.

Keywords: ex-post incentive compatibility, competing mechanisms, quasi ex-post equilibrium, Revelation Principles.

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1 Introduction

This paper studies competing mechanism games where principals announce mechanisms and agents select and communicate with at most one principal. We call it

*Address: Department of Economics, 1280 Main Street West, McMaster University, Hamilton, Ontario, Canada L8S 4M4; Email: hansj@mcmaster.ca

competition in *endogenous exclusive agency*. It is ubiquitous in the literature such as *competing auctions* (Burdett and Sakovics (1999), Coles and Eeckhout (2003), McAfee (1993), Peck (2018), Peters (1997), Peters and Severinov (1997), Virag (2010)), *decentralized matching with contracts* (Azevedo (2014), Burdett, Shi, and Wright (2001), Han (2016), Peters (1991)), *competitive screening* (Rothschild and Stiglitz (1976), Miyazaki (1977)), *directed/competitive search* (Inderst and Wambach (2002), Eeckhout and Kircher (2010), Guerrieri, Shimer, and Wright (2010), Wright, Kircher, Julien, and Guerrieri (2019)), etc. The common feature of these applications is that principals (e.g., sellers, firms) use mechanisms as a way of attracting agents (e.g., buyers, workers) since agents search for a better trading or matching opportunity.

While it is common to fix the set of mechanisms available for each principal in an ad hoc way for equilibrium analysis, an equilibrium allocation depends not only on the solution concept but also on the set of mechanisms allowed in the game. Unfortunately, the standard Revelation Principle used in models with a single principal does not work in competing mechanism games because an agent has not only information on her payoff type but also market information such as principals' mechanisms, their terms of trade, etc.[1] Furthermore, multiplicity of continuation equilibria given a profile of mechanisms may make an equilibrium sensitive to a continuation equilibrium agents play upon a principal's deviation.

This paper constructs a general competition model with no exogenous restrictions on the complexity of mechanisms where most of applications in endogenous exclusive agency are nested, but it is more general. As in most applications, an agent's utility depends on the action chosen by the principal she selects and her payoff type but it can also depend on payoff types of other agents who select the same principal. A principal's utility depends not only on his action but also all agents' payoff types. It can also depend on all principals' actions.

We then propose the solution concept of *robust quasi ex-post equilibrium* by first formulating the notion of quasi ex-post equilibrium and refining it by the robustness to deal with multiplicity of continuation equilibria. In a quasi ex-post equilibrium, an agent's strategy of communicating with every non-deviating principal is ex-post optimal for her upon selecting him, i.e., optimal for her given each possible payoff

[1]We use female pronouns for agents and male pronouns for principals.

type of that agent, each possible set of agents who select the same principal, and each possible payoff type profile of these agents and their communication strategies, and each possible profile of the other principals' mechanisms. We show that in a quasi ex-post equilibrium, the profile of agents' strategies of communicating with each non-deviating principal induces an ex-post incentive compatible (EPIC) direct mechanism from his equilibrium mechanism as a function of the other principals' mechanisms. It means that non-deviating principals punish a deviating principal with EPIC direct mechanisms. All equilibrium analyses in most applications are consistent with the notion of quasi ex-post equilibrium but their equilibria are special cases.[2]

There is multiplicity of continuation equilibria tied to a quasi ex-post equilibrium. For example, consider competition between two principals. Fix a quasi ex-post equilibrium. Suppose that agents' ex-post optimal strategies of communicating with principal 1 induce an EPIC direct mechanism μ_1^E from principal 1's equilibrium mechanism upon principal 2's deviation to a mechanism γ_2 but an EPIC direct mechanism $\acute{\mu}_1^E$ ($\neq \mu_1^E$) from principal 1's equilibrium mechanism upon principal 2's deviation to γ_2'. Because of the ex-post optimality of agents' strategies of communicating with principal 1, μ_1^E can be also induced upon principal 2's deviation to γ_2' in an alternative continuation equilibrium. A quasi ex-post equilibrium is *robust* if each principal cannot gain by his deviation to all possible mechanisms regardless of agents' continuation equilibrium strategies of communicating with him and their selection strategies, conditional on each possible profile of agents' ex-post optimal strategies of communicating with non-deviators.

This paper establishes two revelation principles for a robust quasi ex-post equilibrium. In a quasi ex-post equilibrium, principal $k \neq j$ may change his EPIC direct mechanism as deviating principal changes his mechanism off the path. We show that it does not have to be the case for a robust quasi ex-post equilibrium. It is shown that a robust quasi ex-post equilibrium can be understood in a way that a non-deviating principal punishes a deviating principal with a single EPIC direct mechanism re-

[2]In many applications, there are no informational externalities and a non-deviating principal maintains the same dominant-strategy incentive compatible (DIC) direct mechanism on or off the path following a competing principal's deviation. Some of examples are second-price auctions with reserve price (Burdett and Sakovics (1999), McAfee (1993), Peters (1997), Peters and Severinov (1997), Virag (2010)) and fixed-prices (Burdett, Shi, and Wright (2001), Peters (1984, 1991)). Examples also include Coles and Eeckhout (2003) where the price is conditional on the number of buyers who select the seller.

gardless of the deviating principal's mechanism. This makes it possible to avoid describing a deviating principal's mechanism but allows agents to report only the identity of a deviating principal. The *Weak Revelation Principle* (Theorem 1) shows that an allocation is supported as an equilibrium allocation in a robust quasi ex-post equilibrium if and only if (i) it is an EPIC allocation and (ii) it is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium where each principal's equilibrium mechanism is an EPIC extended direct mechanism in which it is ex-post optimal for the agent to report her true payoff type and the true identity of a deviating principal (if any).

The *Strong Revelation Principle* (Theorem 2) provides the necessary and sufficient condition for an EPIC allocation to be supported in a robust quasi ex-post equilibrium. Due the Weak Revelation Principle, we only need to consider a robust truthful quasi ex-post equilibrium where each principal's equilibrium mechanism is an EPIC extended direct mechanism. Given a profile of EPIC direct mechanisms implemented by non-deviating principals upon principal j's deviation, principal j's direct mechanism is Bayesian incentive compatible (BIC) if and only if there exists a continuation equilibrium strategy of selecting j with which an agent's truthful type reporting to j upon selecting him maximizes her interim utility given that other participating agents report their true types as well. The Strong Revelation Principle shows that an EPIC allocation can be supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if (i) there exists a profile of EPIC extended direct mechanisms that implements the EPIC allocation in the truthful continuation equilibrium on the equilibrium path and (ii) principal j cannot gain in all truthful continuation equilibria upon deviation to any BIC direct mechanism conditional on a profile of EPIC direct mechanisms induced from non-deviating principals' EPIC extended direct mechanisms.

The Strong Revelation Principle provides a way to check if an equilibrium allocation in a specific competition model in applications is a robust quasi ex-post equilibrium allocation. Applying the Strong Revelation Principle, we can show that equilibrium allocations in Cole and Eeckhout (2003) are robust quasi ex-post equilibrium allocations. The robustness of competitive auction distribution shown in Peters (1997) and Han (2015) is indeed a corollary of the Strong Revelation Principle. The crux is that for any of deviating principal j's BIC direct mechanisms μ_j conditional on EPIC direct mechanisms implemented by non-deviators upon j's deviation, there may be multiple continuation equilibrium selection strategies that make μ_j BIC: For the robustness, we choose the selection strategy that maximizes principal j's utility given μ_j. This is the approach that Cole and Eeckhout (2003), Peters (1997), and Han (2015) all take.

The Weak Revelation Principle, in tandem with the Strong Revelation Principle, leads to the characterization of robust quasi ex-post equilibrium allocations in our general model. Because of the Weak Revelation Principle, we consider EPIC allocations that can be supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium where every principal announces EPIC extended direct mechanisms. Let $\rho_k = \{\rho_k^n\}_{n=1}^I$ denote principal k's EPIC extended direct mechanism where ρ_k^n is the EPIC extended direct mechanism used when n agents select principal k. We first consider the set of EPIC extended direct mechanisms, $\mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)$, that share ρ_k^n for $n = 1, 2$. Essentially, in $\mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)$, we fix the way principal k implements EPIC direct mechanisms with one or two participating agents on and also off the path following deviation by each principal $j \neq k$ but there are no restrictions on how to set up ρ_k^n for $n \geq 3$.

Fix $\{\rho_k^1, \rho_k^2\}$ for all k. The threshold of principal j's utility $v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right)$ can be defined as his lowest possible payoff that principals except j can induce by choosing a profile of EPIC extended direct mechanisms from $\mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right) = \{\mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)\}_{k \neq j}$. The ρ^{1-2}-Characterization (Theorem 3) establishes that given $\{\rho_k^1, \rho_k^2\}$ for all k, any EPIC allocation can be supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if it provides the utility for each principal j greater than $v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right)$. It is the constrained characterization.[3] We show that the threshold $v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right)$ is expressed in terms of non-deviating principals' EPIC direct mechanisms and deviating principal j's BIC direct mechanism. Due to the Weak

[3]The proof of the "if" part of the ρ^{1-2}-Characterization is constructive. Given any EPIC allocation that provides a utility for each j greater than $v_j^*(\rho_{-j}^1, \rho_{-j}^2)$, we can associate principal k's EPIC extended direct mechanism $\rho_k^j = \{\rho_k^{jn}\}_{n=1}^I \in \mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)$ that principal k can use in response to each principal j's deviation for all $j \neq k$. We show that from the profile of EPIC extended direct mechanisms $\{\rho_k^j\}_{j \neq k}$, we can construct a single EPIC extended direct mechanism $\bar{\rho}_k = \{\bar{\rho}_k^n\}_{n=1}^I \in \mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)$ that principal k can use for any principal's deviation. This allows us to show that $\{\bar{\rho}_1, \ldots, \bar{\rho}_J\}$ is the profile of equilibrium EPIC extended direct mechanisms that support the EPIC allocation under consideration.

Revelation Principle, the union of all those EPIC allocations over all $\{\rho_k^1, \rho_k^2\}$ for all k is the set of robust quasi ex-post equilibrium allocations independent of ad hoc restrictions on mechanisms allowed in a specific competition model (Theorem 4).

Simpler characterizations are also provided for cases where either (i) only two principals are competing in the model or (ii) a restriction is imposed on preference relations of any pair of agents when a principal selected by them are not selected by any other agent.

2 Related Literature

The central research theme in competing mechanisms is to develop a general tool kit, which, given a solution concept, allows us (a) to check if an equilibrium allocation in a specific competition model survives when a principal is allowed to use more sophisticated mechanisms and (b) to characterize the set of equilibrium allocations in terms of model primitives so that they are independent of ad hoc restrictions imposed on a specific competition model. Our paper achieves this goal with the two Revelation Principles given the solution concept of robust quasi ex-post equilibrium in endogenous exclusive agency.

Endogenous exclusive agency Epstein and Peters (1999) and Attar, Campioni, and Piaser (2018) study competing mechanism games in endogenous exclusive agency. The solution concept in these studies is strongly robust equilibrium which was formally proposed by Han (2007). The strong robustness is stronger than the robustness adopted in our paper in that it requires that no principal gain in *all possible continuation equilibria* upon his deviation to any mechanism.

In their seminal paper, Epstein and Peters (1999) develop the universal language that allows agents to describe any mechanism announced by a competing principal in a model with two principal and two agents where an agent's utility depends only on her payoff type and the action chosen by the principal she selects and a principal's utility depends only on his action and payoff types of two agents. Therefore, there are no action externalities among all players and no informational externalities among agents. They establish a revelation principle, which shows that any equilibrium in a competing mechanism is strongly robust if and only if there exists a payoff-equivalent truthful equilibria where principals offer universal mechanisms. Because of its complexity, it is difficult to apply universal mechanisms in order to check the strong robustness of an equilibrium in a specific model of competition or to characterize the set of strongly robust equilibrium allocations.[4]

Attar, Campioni, and Piaser (2018) focus on how to check the strong robustness of a truthful equilibrium where every principal announces a direct mechanism that is incentive compatible on the equilibrium path. They shows that a truthful equilibrium is strongly robust if no principal can gain in all continuation equilibria in which agents report their true types to a deviating principal upon his deviation to any incentive compatible direct mechanism given non-deviators' direct mechanisms.[5]

In a related example with complete information, Attar, Campioni, Mariotti, and Piaser (2019) show that if exclusive agency relation is rather exogenously given (e.g., competitive hierarchies or exclusive dealing (Martimort (1996), Gal-Or (1997)), not every incentive-feasible allocation in which principals obtain utilities above their minmax over complex mechanisms can be supported in an SPNE. In another example with complete information, two agents, and two principals, they show that there can be an SPNE in which some principal obtains a utility below his minmax utility over direct mechanisms. The finding in this example does not contradict ours because our ρ^{1-2}-Characterization (Theorem 3) is a constrained characterization, conditional on extended direct mechanisms that each non-deviating principal implements on and off the path when one or two agents select him.

Common Agency Agents communicate with all principals in common agency. Three solution concepts are adopted: perfect Bayesian equilibrium (Martimort and Sole (2002), Peters (2001), Yamashita (2010)), strongly robust equilibrium (Han (2007)), Markov equilibrium (Pavan and Calzolari (2009, 2010))

The Menu Theorem plays a central role in studies on perfect Bayesian equilibrium. Martimort and Sole (2002) and Peters (2001) show that in the single common

[4]A mechanism is described by an infinite sequence of real numbers according to the universal language. This reflects the complex nature of mechanisms, called the infinite regress problem. See Section 4 for more discussion.

[5]This implies that we need to check continuation equilibria in which all agents report their true types to a deviating principal upon participation but some agents report false types to a non-deviating principal upon participation.

agency, a menu is strategically equivalent to a complex mechanism if the choices available in the menu are the same as the image set of the complex mechanism under consideration. This gets rid of the agent's communication. The idea of the Menu Theorem is extended to multiple common agency with three or more agents through the recommendation mechanism (Yamashita (2010)). A recommendation mechanism delegates the choice of incentive compatible direct mechanism to agents. If all agents or all but one agent recommend the same direct mechanism, the recommended direct mechanism assigns an action based on agents' type reports submitted along with their recommendations. This approach though does not provide a way to check if an equilibrium in a specific competition model survive when principals are allowed to use more sophisticated mechanisms nor characterize the set of equilibrium allocations in terms of model primitives. The Folk Theorem in Yamashita (2010) expresses the threshold of a principal's payoff only in terms of complex mechanisms allowed in a game.

In a Markov equilibrium (Pavan and Calzolari (2009, 2010)), the agent's behavior depends only on payoff-relevant information. In single common agency, they establish the Markov Revelation Principle that shows a principal can rely on only the extended direct mechanism where the single agent is asked to report her payoff type along with the contracts she chose from the other principals' mechanisms. The Markov Revelation Principle does not lead to the characterization of the set of Markov equilibrium allocations in the single common agency but it is very helpful in deriving a Markov equilibrium allocation in application.

Han (2007) proposes the notion of strong robustness of equilibrium in multiple common agency of complete information: An equilibrium is strongly robust if no principal can gain in all continuation equilibria upon his deviation to any complex mechanism. He establishes sufficient conditions for an equilibrium to be robust. Applying it, he shows that an equilibrium in the game played by agents in Prat and Rustichini (2003) is strongly robust.

Most applications in endogenous exclusive agency are based on models with multiple agents. However, there are relatively a lot more applications in single common agency than those in multiple common agency.[6]

[6]See Martimort (2007) for survey on common agency. Attar, Mariotti, Salanie (2011, 2014), Han (2014), Lavi and Shamashy (2019), Prat and Rustichini (2003) are some of applications in multiple

Other related studies Peters and Trancoso-Valverde (2013) show that any allocation that is implementable by a single mechanism designer (in the sense of Myerson (1979)) can be implementable in a decentralized equilibrium of a variation of the competing mechanism game, where all players are able to offer mechanisms to one another and a second round of communication is available to cross-check messages sent in the first round.

Both endogenous exclusive agency and common agency assume that principals do not observe the other principals' mechanisms, nor can they make his action choice directly conditional on those mechanisms even if they are observable. Peters and Szentes (2012) and Szentes (2015) take an alternative approach in which contracts are observable and all players design contracts that depend directly on the other players' contracts.

3 Preliminaries

Throughout this paper, when a measurable structure is necessary, the corresponding Borel σ - algebra is used. For a set X, $\Delta(X)$ denotes the set of probability distributions on X.

There are J principals. We assume that $J \geq 2$. Each principal j takes an action from a set A_j. We assume that A_j is sufficiently general to include a contingent action, which specifies an action conditional on the set of agents who select principal j. Let $\mathcal{A}_j \equiv \triangle(A_j)$ be the set of (random) actions available for principal j. Let $\mathcal{A} \equiv \mathcal{A}_1 \times \cdots \times \mathcal{A}_J$. There are I agents. We assume that $I \geq 2$ unless specified otherwise. Each agent chooses only one principal. Each agent has payoff-relevant private information, which is parametrized by an element, called the (payoff) type, in Ω. Each agent's type is independently drawn from a probability distribution F on Ω.[7]

Suppose that an agent chooses principal j who is chosen by n agents including herself. Then, the agent's utility is characterized by $u_j^n : \mathcal{A}_j \times \Omega^n \to \mathbb{R}$ and in

common agency.

[7]Most applications take this parsimonious approach. This is consistent with the large market assumption that F is the distribution of agents in the market. It is also the approach that Epstein and Peters (1999) takes. None of the results in our paper depends on this approach. We believe that it is possible to relax this assumption but at the cost of considerable complication of notation.

the most parts of the paper, we assume that principal j's utility is characterized by $v_j^n : \mathcal{A}_j \times \Omega^I \to \mathbb{R}$. These preference relations are a generalization of those specified in Epstein and Peters (1997) in that it allows for any arbitrary number of principals and agents and informational externalities among agents who select the same principal. As in Epstein and Peters (1997), we also allow a principal's utility to depend on all agents' payoff types. If an agent does not choose any principal, her reservation utility is $\underline{u}(\omega)$, where ω is the agent's payoff type in Ω. If no agent chooses principal j, his reservation utility is $\underline{v}_j$.

Most applications in endogenous exclusive agency in the literature satisfy the preference relations specified above. For example, in *competing auctions*, a buyer's utility only depends on the allocation and monetary transfers determined by the seller she selects and potentially the payoff types of buyers who select the same seller; A seller's profit depends on his allocation and monetary transfers but also potentially on the payoff types of buyers who select him or all buyers. In *matching problems*, an individual's matching utility only depends on her and her matching partners' characteristics (and possibly transfers among them), but not on characteristics of individuals in other matches. In *competitive screening*, an insured or worker's underlying characteristics and her insurance company or employer's decision affect both parties' utilities. Most of applications in *directed/competitive search* also satisfy these preference relations.

All the results in our paper also work with proper modifications even when action externalities among all principals are allowed so that a principal's utility depends on all principals' actions as well as all agents' payoff types. (See Appendix A.1 for how to extend our result in the case of action externalities among principals).

For each number n of participating agents, principal j's mechanism specifies his (random) action in $\mathcal{A}_j$ as a function of messages from participating agents, $\gamma_j^n : (M_j)^n \to \mathcal{A}_j$, where M_j is the set of messages that a participating agent can send. Principal j's mechanism γ_j consists of a set of functions, $\gamma_j = \left\{\gamma_j^n\right\}_{n=1}^I$. Let Γ_j be the set of all possible mechanisms available for principal j. Let $\Gamma \equiv \Gamma_1 \times \cdots \times \Gamma_J$.

The competing mechanism game starts when each principal j announces a mechanism from Γ_j. After observing a profile of mechanisms, each agent simultaneously selects only one principal and sends a message to him. The profile of messages sent by participating agents determines the principal's action and payoffs are realized.

Each profile of mechanisms determines a subgame played by agents and we adopt the notion of Bayes Nash equilibrium for the solution concept for a continuation equilibrium of the subgame. For simplicity of notation, we look for a symmetric continuation equilibrium where agents employ the same strategies. Let $c_j : \Gamma \times \Omega \to M_j$ be each agent's communication strategy upon selecting principal j.[8] Let $\pi_j : \Gamma \times \Omega \to [0,1]$ denote each agent's strategy of selecting principal j. It specifies probability with which an agent selects principal j as a function of a mechanism profile and her type. Let $\{c, \pi\} \equiv \{(c_1, \ldots, c_J), (\pi_1, \ldots, \pi_J)\}$ be a continuation equilibrium if no agent has any incentive to deviate from either the communication strategy c or the selection strategy π for any of her type and any profile of mechanisms offered by principals.

Fix a continuation equilibrium $\{c, \pi\}$. Given $\gamma = (\gamma_1, \ldots, \gamma_J) \in \Gamma$, the probability that each agent does not select principal j is equal to

$$z_{\pi_j}(\gamma) = 1 - \int_\Omega \pi_j(\gamma, x) dF(x)$$

Given $\gamma = (\gamma_1, \ldots, \gamma_J) \in \Gamma$, suppose that n agents participate in γ_j. When $(\omega_1, \ldots, \omega_n)$ is the type profile of the n agents, the agent of type ω_1 has the ex-post utility of

$$u_j^n\left(\gamma_j^n\left[c_j(\gamma, \omega_1), \cdots, c_j(\gamma, \omega_n)\right], (\omega_1, \ldots, \omega_n)\right).$$

Given a profile of mechanisms γ and the selection of principal j, (c_j, π_j) yields the interim utility for the agent of type ω_1 as follows.

$$\begin{aligned} U_j(\gamma, c_j, \pi_j, \omega_1) = \sum_{n=0}^{I-1} \binom{I-1}{n} z_{\pi_j}(\gamma)^{I-1-n} \\ \times \int_\Omega \cdots \int_\Omega u_j^n\left(\gamma_j^n\left[c_j(\gamma_j, \omega_1), \cdots, c_j(\gamma_j, \omega_n)\right], (\omega_1, \ldots, \omega_n)\right) \\ \times \pi_j(\gamma, \omega_1) dF(\omega_1) \times \cdots \times \pi_j(\gamma, \omega_n) dF(\omega_n). \quad (1) \end{aligned}$$

[8] Note that c_j is a pure communication strategy. We can also allow for a mixed communication strategy. Since a mechanism assigns a random action conditional on participating agents' messages, allowing for mixed communication strategies does not alter any of our results. We employ pure communication strategies for simplicity of notation.

Finally, the interim utility of the agent of type $\omega_1 \in \Omega$ associated with a continuation equilibrium $\{c, \pi\}$ is

$$U(\gamma, c, \pi, \omega_1) = \sum_{j=1}^{J} \pi_j(\gamma, \omega_1) U_j(\gamma, c_j, \pi_j, \omega_1) + \left(1 - \sum_{j=1}^{J} \pi_j(\gamma, \omega_1)\right) \underline{u}(\omega_1).$$

Given $\gamma = \left(\gamma_j, \gamma_{-j}\right)$, the ex-ante utility for principal j associated with a continuation equilibrium $\{c, \pi\}$ is

$$\begin{aligned} V_j(\gamma_j, \gamma_{-j}, c_j, \pi_j) = \sum_{n=1}^{I} \left(\begin{array}{c} I \\ n \end{array} \right) \\ \times \int_\Omega \cdots \int_\Omega v_j^n \left(\gamma_j^n \left[c_k(\gamma_j, \gamma_{-j}, \omega_1), \cdots, c_k(\gamma_j, \gamma_{-j}, \omega_n)\right], (\omega_1, \ldots, \omega_I)\right) \\ \times \pi_j(\gamma_j, \gamma_{-j}, \omega_1) dF(\omega_1) \times \cdots \times \pi_j(\gamma_j, \gamma_{-j}, \omega_n) dF(\omega_n) \\ \times \left[1 - \pi_j(\gamma_j, \gamma_{-j}, \omega_{n+1})\right] dF(\omega_{n+1}) \times \cdots \times \left[1 - \pi_j(\gamma_j, \gamma_{-j}, \omega_I)\right] dF(\omega_I) \\ + z_{\pi_j}\left(\gamma_j, \gamma_{-j}\right)^I \times \underline{v}_j. \quad (2) \end{aligned}$$

Definition 1 *$\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ is a* (perfect Bayesian) equilibrium *if $\{\tilde{c}, \tilde{\pi}\}$ is a continuation equilibrium and, for $j \in \mathcal{J} \equiv \{1, \ldots, J\}$ and all $\gamma_j \in \Gamma_j$*

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq V_j(\gamma_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j).$$

3.1 Allocations

Principal j's direct mechanism is $\mu_j = \left\{\mu_j^n\right\}_{n=1}^{I}$ with $\mu_j^n : \Omega^n \to \mathcal{A}_j$. For all $j \in \mathcal{J}$, let $\phi_j : \Omega \to \Omega$ be the identity function. $\mu \equiv (\mu_1, \ldots, \mu_J)$ is a profile of Bayesian incentive compatible (BIC) direct mechanism if there exists $\pi = (\pi_1, \ldots, \pi_J)$ such that $\{\phi, \pi\} \equiv \{(\phi_1, \ldots, \phi_J), (\pi_1, \ldots, \pi_J)\}$ constitutes a continuation equilibrium at μ. Let $\mathcal{D}^B$ be the set of all possible BIC profiles of mechanisms. Let $\{\mu, \pi\}$ be an allocation if $\{\phi, \pi\}$ constitutes a continuation equilibrium at μ.

The Bayesian incentive compatibility of μ_j is based on an agent's interim utility, which in turn depends on not only the probability distribution on agents' types but also agents' endogenous selection strategy π. Because agents' selection strategy π depends on a profile of direct mechanisms offered by all principals, $\mathcal{D}^B$ is not a

Cartesian product.

The ex-post incentive compatibility is based on an agent's ex-post utility. $\mu_j = \{\mu_j^n\}_{n=1}^I$ is ex-post incentive compatible (EPIC) if for all $n \in \{1, \dots I\}$, all $(\omega_1, \dots, \omega_n) \in \Omega^n$, all $i \in \{1, \dots, n\}$, and all $\omega_i' \in \Omega$, the ex-post utilities for the agent of type ω_i satisfy

$$u_j^n\left(\mu_j^n\left(\omega_1 \dots \omega_n\right), (\omega_1, \dots, \omega_n)\right) \geq u_j^n\left(\mu_j^n\left(\omega_i', (\omega_h)_{h \neq i}\right), (\omega_1, \dots, \omega_n)\right).$$

Let $\mathcal{D}_j^{En}$ be the set of all EPIC direct mechanisms for principal j when n agents select principal j. The set of all possible EPIC direct mechanisms for principal j is defined as

$$\mathcal{D}_j^E \equiv \left\{\mu_j = \{\mu_j^n\}_{n=1}^I : \mu_j^n \in \mathcal{D}_j^{En}\ \forall n\right\}.$$

The ex-post incentive compatibility is independent of agents' selection strategy. The set of all possible profiles of EPIC direct mechanisms $\mathcal{D}^E$ is a Cartesian product, i.e., $\mathcal{D}^E \equiv \mathcal{D}_1^E \times \cdots \times \mathcal{D}_J^E$.

4 Quasi Ex Post Equilibrium

In an equilibrium, an agent's strategies are interim optimal for her. In an quasi ex-post equilibrium, an agent's strategy of communicating with a non-deviating principal is ex-post optimal. A quasi ex-post equilibrium is defined as follows.

Definition 2 $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ *is a* quasi ex-post equilibrium *if it is an equilibrium and, for all $k \in \mathcal{J}$, all $\gamma_{-k} \in \Gamma_{-k}$, all $n \in \{1, \dots, I\}$, all $(\omega_1, \dots, \omega_n) \in \Omega^n$, all $i \in \{1, \dots, n\}$, all $m_i \in M_j$, the ex-post utilities for the agent of type ω_i satisfy*

$$u_k^n\left(\tilde{\gamma}_k^n\left[\tilde{c}_k(\tilde{\gamma}_k, \gamma_{-k}, \omega_i), \left(\tilde{c}_k(\tilde{\gamma}_k, \gamma_{-k}, \omega_h)\right)_{h \neq i}\right], (\omega_1, \dots, \omega_n)\right) \geq \\ u_k^n\left(\tilde{\gamma}_k^n\left[m_i, \left(\tilde{c}_k(\tilde{\gamma}_k, \gamma_{-k}, \omega_h)\right)_{h \neq i}\right], (\omega_1, \dots, \omega_n)\right). \quad (3)$$

Consider an agent's communication with any non-deviating principal k upon selecting him when the other principals' mechanisms are γ_{-k}. (3) compares an agent's ex-post utilities when she chooses principal j and her type is ω_i in the type profile

$(\omega_1,\ldots,\omega_n)$ of n participating agents. We fix any other participating agent h's continuation equilibrium strategy of communicating with principal j as $\tilde{c}_k(\tilde{\gamma}_k,\gamma_{-k},\omega_h)$. (3) shows that it is ex-post optimal for the agent of type ω_i to follow her continuation equilibrium strategy $\tilde{c}_k(\tilde{\gamma}_k,\gamma_{-k},\omega_i)$ instead of sending any other message m_i. Ex-post optimality implies interim optimality but not the other way around.

$\tilde{c}_k(\tilde{\gamma}_k,\gamma^i_{-k},\omega_i)$ denotes the message the agent of type ω_i sends to principal k upon selecting k when k offers the equilibrium mechanism $\tilde{\gamma}_k$, whereas the other principals offer $\gamma^i_{-k}\in\Gamma_{-k}$. Define $\tilde{g}_k=\{\tilde{g}^n_k\}^I_{n=1}$ such that for all $n\in\{1,\ldots,I\}$, all $(\omega_1,\ldots,\omega_n)\in\Omega^n$, all $\left(\gamma^1_{-k},\ldots,\gamma^n_{-k}\right)\in(\Gamma_{-k})^n$

$$\tilde{g}^n_k\left[\left(\gamma^1_{-k},\omega_1\right),\ldots,\left(\gamma^n_{-k},\omega_n\right)\right]\equiv\tilde{\gamma}^n_k\left[\tilde{c}_k(\tilde{\gamma}_k,\gamma^1_{-k},\omega_1),\ldots,\tilde{c}_k(\tilde{\gamma}_k,\gamma^n_{-k},\omega_n)\right]. \tag{4}$$

Therefore, $\tilde{g}^n_k$ is a mapping from $(\Gamma_{-k}\times\Omega)^n$ into $\mathcal{A}_k$. $\tilde{g}_k=\{\tilde{g}^n_k\}^I_{n=1}$ can be thought of as an *extended direct mechanism* in which each participating agent is asked to report competing principals' mechanisms and her type in $\Gamma_{-k}\times\Omega$.

Lemma 1 *Given a quasi ex-post equilibrium $\{\tilde{\gamma},\tilde{c},\tilde{\pi}\}$, $\tilde{g}_k=\{\tilde{g}^n_k\}^I_{n=1}$ is* EPIC *over $\Gamma_{-k}\times\Omega$, that is for all $k\in\mathcal{J}$, all $\gamma_{-k},\gamma'_{-k}\in\Gamma_{-k}$, all $n\in\{1,\ldots,I\}$, all $(\omega_1,\ldots,\omega_n)\in\Omega^n$, all $i\in\{1,\ldots,n\}$, and all $\omega'_i\in\Omega$,*

$$u^n_k\left(\tilde{g}^n_k\left[\left(\gamma_{-k},\omega_i\right),\left(\gamma_{-k},\omega_h\right)_{h\neq i}\right],(\omega_1,\ldots,\omega_n)\right)\geq u^n_k\left(\tilde{g}^n_k\left[\left(\gamma'_{-k},\omega'_i\right),\left(\gamma_{-k},\omega_h\right)_{h\neq i}\right],(\omega_1,\ldots,\omega_n)\right).$$

Proof. This is the direct implication of (3) given the definition of $\tilde{g}_k$ in (4). ■

With slight abuse of notation, given a quasi ex-post equilibrium $\{\tilde{\gamma},\tilde{c},\tilde{\pi}\}$, define $\tilde{g}_k\left(\gamma_j\right)=\left\{\tilde{g}^n_k\left(\gamma_j\right)\right\}^I_{n=1}$ for all $j\in\mathcal{J}$, all $\gamma_j\in\Gamma_j$, and all $k\neq j$ as

$$\tilde{g}^n_k\left(\gamma_j\right)\equiv\tilde{g}^n_k\left[\left(\gamma_j,(\tilde{\gamma}_\ell)_{\ell\neq k,j},\cdot\right),\ldots,\left(\gamma_j,(\tilde{\gamma}_\ell)_{\ell\neq k,j},\cdot\right)\right]\text{ for all }n\in\{1,\ldots,I\} \tag{5}$$

Corollary 1 *Given a quasi ex-post equilibrium $\{\tilde{\gamma},\tilde{c},\tilde{\pi}\}$, $\tilde{g}^n_k\left(\gamma_j\right)$ is an EPIC direct mechanism for each $n\in\{1,\ldots,I\}$ and each $\gamma_j\in\Gamma_j$*

Proof. This directly follows the ex-post incentive compatibility of the extended direct mechanism $\tilde{g}_k=\{\tilde{g}^n_k\}^I_{n=1}$. ■

Essentially, each non-deviating principal k punishes principal j with the EPIC direct mechanism $\tilde{g}_k\left(\gamma_j\right)$ upon j's deviation to γ_j. Therefore, it is possible that $\tilde{g}_k\left(\gamma_j\right) \neq \tilde{g}_k\left(\gamma_j'\right)$ if $\gamma_j \neq \gamma_j'$, that is, principal k changes his EPIC direct mechanism if principal j changes a mechanism that he uses upon his deviation. This is why principal k wants to know the mechanism principal j deviates to in a quasi ex-post equilibrium. While $\tilde{g}_k = \{\tilde{g}_k^n\}_{n=1}^I$ does such a job, its message space used in describing others' mechanisms is Γ_{-k}, which may be specific to the competition model embodied in a competing mechanism game.

For a strongly robust equilibrium, Epstein and Peters (1999) develop the universal language independent of any ad hoc restrictions in a set of mechanisms allowed in a competing mechanism game. The universal mechanism allows agents to describe any mechanism that a competing principal may deviate to with this universal language.

An action determined by my mechanism depends on mechanisms offered by my opponents, their mechanisms depend in turn my mechanism, and so on. This is called the infinite regress problem and it is outwardly similar to the hierarchies of beliefs (e.g., Brandenburger and Dekel (1993), Mertens and Zamir (1985)). This makes the universal language too complicated for applications (A typical element in the universal language is an infinite sequence of real numbers).

We shall show that for a robust quasi ex-post equilibrium, it is sufficient for agents to reveal only the identity of a deviating principal (if any) in terms of market information they should deliver. Not only is it independent of any ad hoc restrictions in a set of mechanisms allowed in a specific competition model but it is free from the infinite regress problem. In the next section, we define the robustness of a quasi ex-post equilibrium.

5 Robust Quasi Ex Post Equilibrium

For the robustness we adopt for a quasi ex-post equilibrium, we start with the multiplicity of continuation equilibria that is tied to the notion of quasi ex-post equilibrium. Fix a quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. Suppose that principal j deviates to $\gamma_j \neq \tilde{\gamma}_j$ given a quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. It is ex-post optimal for an agent to follow $\tilde{c}_k(\gamma_j, \tilde{\gamma}_{-j}, \cdot)$ for communication with non-deviating principal $k \neq j$. It is then also ex-post optimal for her to follow $\tilde{c}_k(\gamma_j, \tilde{\gamma}_{-j}, \cdot)$ for communication with the same non-deviating principal $k \neq j$ when principal j deviates to any other mechanism $\gamma'_j \neq \gamma_j, \tilde{\gamma}_j$.

Lemma 2 *Fix a quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. For all $j \in \mathcal{J}$, all $k \neq j$, all $\gamma_j, \gamma'_j \in \Gamma_j$,*

$$c'_k(\gamma'_j, \tilde{\gamma}_{-j}, \cdot) = \tilde{c}_k(\gamma_j, \tilde{\gamma}_{-j}, \cdot) \tag{6}$$

is ex-post optimal for an agent.

Proof. Fix γ_j. Given (6), (3) leads to the following inequality: For all $\gamma'_j \in \Gamma_j$, all $n \in \{1, \ldots, I\}$, all $(\omega_1, \ldots, \omega_n) \in \Omega^n$, all $m_i \in M_k$,

$$u_k^n \left(\tilde{\gamma}_k^n \left[c'_k(\gamma'_j, \tilde{\gamma}_{-j}, \omega_i), \left(c'_k(\gamma'_j, \tilde{\gamma}_{-j}, , \omega_h) \right)_{h \neq i} \right], (\omega_1, \ldots, \omega_n) \right)$$
$$\geq u_k^n \left(\tilde{\gamma}_k^n \left[m_i, \left(c'_k(\gamma'_j, \tilde{\gamma}_{-j}, \omega_h) \right)_{h \neq i} \right], (\omega_1, \ldots, \omega_n) \right),$$

which shows that $c'_k(\tilde{\gamma}_{-j}, \gamma'_j, \cdot)$ is ex-post optimal for an agent. ∎

The observation in Lemma 2 leads to the following robustness of a quasi ex-post equilibrium.

Definition 3 *A quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ is* robust *if for all $j \in \mathcal{J}$, all $\gamma_j, \gamma'_j \neq \tilde{\gamma}_j$*

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq V_j(\gamma'_j, \tilde{\gamma}_{-j}, c'_j, \pi'_j)$$

for all $\left(c'_j, \pi'_j\right)$ such that there exists π'_{-j} that makes $\left(c'_j, c'_{-j}, \pi'_j, \pi'_{-j}\right)$ a continuation equilibrium at $\left(\gamma'_j, \tilde{\gamma}_{-j}\right)$, with c'_{-j} satisfying (6) for all $k \neq j$ given γ_j.

Given a quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$, let us define $\tilde{G}_k^j$ for all $j \in \mathcal{J}$ and all $k \neq j$,

$$\tilde{G}_k^j \equiv \left\{ \tilde{g}_k\left(\gamma_j\right) : \gamma_j \in \Gamma_j, \gamma_j \neq \tilde{\gamma}_j \right\},$$

which is the set of all EPIC direct mechanisms that $\tilde{c}_k(\gamma_j, \tilde{\gamma}_{-j}, \cdot)$ induces from $\tilde{\gamma}_k$ upon principal j's deviation to all $\gamma_j \neq \tilde{\gamma}_j$. Let $\tilde{G}_{-j}^j \equiv \left(\tilde{G}_k^j\right)_{k \neq j}$. Let $\mathcal{M}_j^B\left(\mu_{-j}^E\right)$ be the set of all BIC direct mechanisms conditional on μ_{-j}^E. Let $\Pi_j\left(\mu_j, \mu_{-j}^E\right)$ be the set of all continuation equilibrium strategies of selecting principal j conditional on truthful type reporting to every principal upon selecting them when μ_{-j}^{E} are the mechanisms of principals except j and $\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^{E}\right)$ is principal j's mechanism.

Proposition 1 *A quasi ex-post equilibrium* $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ *is robust if and only if*

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq \sup_{\mu_{-j}^{E} \in \tilde{G}_{-j}^{j}} \sup_{\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^{E}\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \mu_{-j}^{E}\right)} V_j(\mu_j, \mu_{-j}^{E}, \phi_j, \pi_j). \quad (7)$$

Proof. Fix a quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. $\tilde{G}_{-j}^{j}$ is the set of all EPIC direct mechanisms that $\tilde{c}_{-j}$ induces from $\tilde{\gamma}_{-j}$ upon principal j's deviation to all $\gamma_j \neq \tilde{\gamma}_j$. Let

$$\mathcal{E}_j\left(\gamma_j', \mu_{-j}^{E}\right) \equiv \left\{\left(c_j', \pi_j'\right) : \left(c_j', \phi_{-j}, \pi_j', \pi_{-j}'\right) \text{ is a cont. eq. at } \left(\gamma_j', \mu_{-j}^{E}\right) \text{ for some } \pi_{-j}'\right\}.$$

The robustness defined in Definition 3 implies that $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ is robust if and only if, for all $\gamma_j' \neq \tilde{\gamma}_j$ and all $\mu_{-j}^{E} \in \tilde{G}_{-j}^{j}$

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq \sup_{\left(c_j', \pi_j'\right) \in \mathcal{E}_j\left(\gamma_j', \mu_{-j}^{E}\right)} V_j(\gamma_j', \mu_{-j}^{E}, c_j', \pi_j'). \quad (8)$$

For each $\gamma_j' = \left\{\acute{\gamma}_j^n\right\}_{n=1}^{I} \neq \tilde{\gamma}_j$, each $\mu_{-j}^{E} \in \tilde{G}_{-j}^{j}$, and each $\left(c_j', \pi_j'\right) \in \mathcal{E}_j\left(\gamma_j', \mu_{-j}^{E}\right)$, let us define $\tau_j\left(\gamma_j', c_j', \mu_{-j}^{E}\right) = \left\{\tau_j^n\left(\gamma_j', c_j', \mu_{-j}^{E}\right)\right\}_{n=1}^{I}$ as, for all $n \in \{1, \ldots, I\}$, and all $(\omega_1, \ldots, \omega_n) \in \Omega^n$

$$\tau_j^n\left(\gamma_j', c_j', \mu_{-j}^{E}\right)(\omega_1, \ldots, \omega_n) \equiv \acute{\gamma}_j^n\left(c_j'\left(\gamma_j', \mu_{-j}^{E}, \omega_1\right), \ldots, c_j'\left(\gamma_j', \mu_{-j}^{E}, \omega_n\right)\right).$$

Because c_j' is part of a continuation equilibrium in $\mathcal{E}_j\left(\gamma_j', \mu_{-j}^{E}\right)$, $\tau_j\left(\gamma_j', c_j', \mu_{-j}^{E}\right)$ is BIC and belongs to $\mathcal{M}_j^B\left(\mu_{-j}^{E}\right)$. Furthermore, principal j can directly deviate to any μ_j in $\mathcal{M}_j^B\left(\mu_{-j}^{E}\right)$, followed by agents' truthful type reporting to every principal upon participation, given μ_{-j}^{E}. Therefore, we have that for all $\mu_{-j}^{E} \in \tilde{G}_{-j}^{j}$

$$\sup_{\left(c_j', \pi_j'\right) \in \mathcal{E}_j\left(\gamma_j', \mu_{-j}^{E}\right)} V_j(\gamma_j', \mu_{-j}^{E}, c_j', \pi_j') = \sup_{\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^{E}\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \mu_{-j}^{E}\right)} V_j(\mu_j, \mu_{-j}^{E}, \phi_j, \pi_j). \quad (9)$$

(8) and (9) lead to (7). ■

6 Revelation Principles

We propose an EPIC extended direct mechanism that asks each participating agent to report her type and the identity of a deviating principal if any. In this extended direct mechanism, reporting k to principal k means that no principal has deviated.

Definition 4 *An extended direct mechanism* $\rho_k = \{\rho_k^n\}_{n=1}^I$ *with* $\rho_k^n : (\mathcal{J} \times \Omega)^n \to \mathcal{A}_k$ *for all* $n \in \{1, \ldots, I\}$ *for principal* k *is ex-post incentive compatible over* $\mathcal{J} \times \Omega$ *if, for all* $n \in \{1, \ldots, I\}$, *all* $j \in \mathcal{J}$, *all* $(\omega_1, \ldots, \omega_n) \in \Omega^n$, *all* $i \in \{1, \ldots, n\}$, *all* $(\ell, \omega_i') \in \mathcal{J} \times \Omega$, *the ex-post utilities of the agent of type* ω_i *satisfy*

$$u_k^n\left(\rho_k^n\left[(j, \omega_i), (j, \omega_h)_{h \neq i}\right], (\omega_1, \ldots, \omega_n)\right) \geq u_k^n\left(\rho_k^n\left[(\ell, \omega_i'), (j, \omega_h)_{h \neq i}\right], (\omega_1, \ldots, \omega_n)\right).$$

Let Λ_k^n be the set of all EPIC extended direct mechanisms $\rho_k^n : (\mathcal{J} \times \Omega)^n \to \mathcal{A}_k$ for principal k when there are n participating agents.

Lemma 3 *For all* $k \in \mathcal{J}$, *an extended direct mechanism* $\rho_k = \{\rho_k^n\}_{n=1}^I$ *is EPIC if and only if* ρ_k^n *is EPIC for all* $n \in \{1, \ldots, I\}$.

The proof of Lemma 3 is straightforward, so it is omitted. Because of Lemma 3, the set of all EPIC extended direct mechanisms for principal k is defined as

$$\Lambda_k \equiv \left\{\rho_k = \{\rho_k^n\}_{n=1}^I : \rho_k^n \in \Lambda_k^n \ \forall n\right\}.$$

We assume that Γ_k is sufficiently large to include ρ_k (or a mechanism homeomorphic to it) so that $\Lambda_k \subset \Gamma_k$. With slight abuse of notation, let $\rho_k(j) = \{\rho_k^n(j)\}_{n=1}^I$ be a direct mechanism such that

$$\rho_k^n(j) \equiv \rho_k^n((j, \cdot), \ldots, (j, \cdot)) \in \mathcal{D}_k^{En}.$$

Note that $\rho_k^n(j)$ is EPIC over payoff types for all $n \in \{1, \ldots, I\}$.

We first focus on a truthful equilibrium where each principal j offers an EPIC extended direct mechanism in Λ_j and agents report their true types and the true identity of a deviating principal (if any) upon participation. Let $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ be a truthful equilibrium in which $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ is a profile of EPIC extended direct mechanisms.

Clearly, a truthful equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is a quasi ex-post equilibrium. We establish the necessary and sufficient condition for a truthful equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ to be robust.

Corollary 2 *A truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is robust if and only if, for all $j \in \mathcal{J}$*

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi_j, \pi_j\right), \tag{10}$$

where $\bar{\rho}_{-j}(j) = (\bar{\rho}_k(j))_{k \neq j}$.

Proof. This follows from Proposition 1. Note that $\tilde{G}^j_{-j}$ in (7) is the set of all profiles of EPIC direct mechanisms that are induced by the profile of communication strategies $\tilde{c}_{-j}$ for principals except j upon principal j's deviation to all possible $\gamma_j \neq \tilde{\gamma}_j$ in a quasi ex post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. In a truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$, $\bar{c}_{-j}$ always induces the unique profile of EPIC direct mechanisms, $\bar{\rho}_{-j}(j)$, regardless of the mechanism that principal j deviates to. In other words, $\tilde{G}^j_{-j}$ is a singleton set and it only includes $\bar{\rho}_{-j}(j)$. For a truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$, (7) therefore becomes (10). ■

Now we establish the Weak Revelation Principle.

Theorem 1 (Weak Revelation Principle) *An allocation is supported as an equilibrium allocation in a robust quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ if and only if it is an EPIC allocation and it is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$.*

Proof. In a quasi ex-post equilibrium, agents' strategies of communicating with non-deviating principals are ex-post optimal. It implies that if no principal deviates, the allocation implemented on the path is EPIC. Therefore, we only focus on an EPIC allocation for an allocation supportable in a robust quasi ex-post equilibrium.

The proof of the "if" part is straightforward because any robust truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is a robust quasi ex-post equilibrium.

For the proof of the "only if" part, we start with an EPIC allocation $\left\{\mu^E, \pi\right\}$ that is supported as an equilibrium allocation in a robust quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$.

For all $k, j \in \mathcal{J}$,

$$\beta_j(k) \equiv \begin{cases} \tilde{\gamma}_j & \text{if } j = k \\ \gamma_j^\circ & \text{otherwise} \end{cases},$$

where γ_j° is an arbitrary mechanism in Γ_j. For all $k \in \mathcal{J}$, we construct $\bar{\rho}_k = \{\bar{\rho}_k^n\}_{n=1}^I$ as follows. For all $n \in \{1, \ldots, I\}$, all $((j_1, \omega_1), \ldots, (j_n, \omega_n)) \in (\mathcal{J} \times \Omega)^n$,

$$\bar{\rho}_k^n((j_1, \omega_1), \ldots, (j_n, \omega_n)) = \tilde{\gamma}_k^n\left[\tilde{c}_k(\beta_{j_1}(k), \tilde{\gamma}_{-j_1}, \omega_1), \ldots, \tilde{c}_k(\beta_{j_n}(k), \tilde{\gamma}_{-j_n}, \omega_n)\right]. \quad (11)$$

Because $\tilde{c}_k$ is ex-post optimal for an agent upon selecting $\tilde{\gamma}_k$ on the path or off the path following any other principal's unilateral deviation, $\bar{\rho}_k$ is EPIC. Furthermore, $\bar{\rho}_{-j}(j) \in \tilde{G}_{-j}^j$.

We construct the profile of agents' communication and selection strategies for a robust truthful quasi ex-post equilibrium. If no principal j deviates from $\bar{\rho}_j$, then we choose $\bar{c}_j$ and $\bar{\pi}_j$ such that for all $j \in \mathcal{J}$, all $i \in \{1, \ldots, I\}$, all $\omega_i \in \Omega$

$$\bar{c}_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \omega_i\right) = (j, \omega_i), \quad (12)$$
$$\bar{\pi}_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \omega_i\right) = \tilde{\pi}_j\left(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \omega_i\right). \quad (13)$$

Because $\bar{\rho}_j$ is EPIC, $\bar{c}_j$ in (12) is ex-post optimal for each agent. (11) and (12) lead to

$$\bar{\rho}_j(j) = \left\{\bar{\rho}_j^n(j)\right\}_{n=1}^I = \left\{\tilde{\gamma}_j^n\left[\tilde{c}_k(\tilde{\gamma}, \cdot), \ldots, \tilde{c}_k(\tilde{\gamma}, \cdot)\right]\right\}_{n=1}^I. \quad (14)$$

(12) - (14) show that the continuation equilibrium $\{\bar{c}, \bar{\pi}\}$ at $\left(\bar{\rho}_j, \bar{\rho}_{-j}\right)$ preserves the EPIC equilibrium allocation supported in a robust quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ and hence we have that for all $j \in \mathcal{J}$,

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) = V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j). \quad (15)$$

Suppose that principal j deviates to $\gamma_j' \neq \bar{\rho}_j$. To construct a continuation equilibrium upon principal j's deviation to $\gamma_j' \neq \bar{\rho}_j$, we take advantage of the robustness of the original quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$. Given $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$, we pick a continuation equilibrium $\left(c'_j, c'_{-j}, \pi'_j, \pi'_{-j}\right)$ at $\left(\gamma'_j, \tilde{\gamma}_{-j}\right)$ such that

$$c'_{-j}\left(\gamma'_j, \tilde{\gamma}_{-j}, \cdot\right) = \tilde{c}_{-j}\left(\gamma^{\circ}_j, \tilde{\gamma}_{-j}, \cdot\right). \tag{16}$$

Because $\tilde{c}_{-j}\left(\tilde{\gamma}_{-j}, \gamma^{\circ}_j, \cdot\right)$ is ex-post optimal for agents to communicate with non-deviators upon j's deviation to γ°_j, $c'_{-j}\left(\tilde{\gamma}_{-j}, \gamma'_j, \cdot\right)$ is also ex-post optimal for agents to communicate with non-deviators upon j's deviation to γ'_j (Lemma 2). Because of the robustness of the quasi ex-post equilibrium $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$, we have that for all $j \in \mathcal{J}$,

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq V_j(\gamma'_j, \tilde{\gamma}_{-j}, c'_j, \pi'_j). \tag{17}$$

We choose a continuation equilibrium $\{\bar{c}, \bar{\pi}\}$ at $\left(\gamma'_j, \bar{\rho}_{-j}\right)$ such that for all $i \in \mathcal{I}$ and all $\omega_i \in \Omega$,

$$\begin{aligned}
\bar{c}_j\left(\gamma'_j, \bar{\rho}_{-j}, \omega_i\right) &= c'_j\left(\gamma'_j, \tilde{\gamma}_{-j}, \omega_i\right), &(18)\\
\bar{c}_k\left(\gamma'_j, \bar{\rho}_{-j}, \omega_i\right) &= (j, \omega_i), \forall k \neq j, &(19)\\
\bar{\pi}_j\left(\gamma'_j, \bar{\rho}_{-j}, \omega_i\right) &= \pi'_j\left(\gamma'_j, \tilde{\gamma}_{-j}, \omega_i\right), &(20)\\
\bar{\pi}_k\left(\gamma'_j, \bar{\rho}_{-j}, \omega_i\right) &= \pi'_k\left(\gamma'_j, \tilde{\gamma}_{-j}, \omega_i\right), \forall k \neq j &(21)
\end{aligned}$$

Given the EPIC extended direct mechanisms $\bar{\rho}_{-j}$, (18) - (21) imply that for all $j \in \mathcal{J}$,

$$V_j\left(\gamma'_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) = V_j(\gamma'_j, \tilde{\gamma}_{-j}, c'_j, \pi'_j). \tag{22}$$

(15), (17), and (22) together imply that for all $j \in \mathcal{J}$ and all $\gamma'_j \neq \bar{\rho}_j$,

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) \geq V_j\left(\gamma'_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right),$$

which shows that $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is a truthful quasi ex-post equilibrium. For the robustness of $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$, note that the robustness of $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ implies that (7) in Proposition 1 is satisfied. Because $\tilde{g}_{-j}\left(\gamma^{\circ}_j\right) \in \tilde{G}^j_{-j}$, (7) implies that

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}_j, \tilde{\pi}_j) \geq \sup_{\mu_j \in \mathcal{M}^B_j\left(\tilde{g}_{-j}\left(\gamma^{\circ}_j\right)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \tilde{g}_{-j}\left(\gamma^{\circ}_j\right)\right)} V_j(\mu_j, \tilde{g}_{-j}\left(\gamma^{\circ}_j\right), \phi_j, \pi_j). \tag{23}$$

Because $\bar{\rho}_{-j}(j) = \tilde{g}_{-j}\left(\gamma^{\circ}_j\right)$, (23) satisfies (10) in Corollary 2 and therefore $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is robust. ■

The Weak Revelation Principle establishes that for an allocation supportable in a robust quasi ex-post equilibrium, there is no loss of generality to focus on a robust truthful quasi ex-post equilibrium where each principal j announces an EPIC extended direct mechanism in Λ_j, whereas they can still deviate to any other mechanisms if they want. The following theorem establishes the Strong Revelation Principle for a robust truthful quasi ex-post equilibrium.

Theorem 2 (Strong Revelation Principle) *Any EPIC allocation $\left\{\mu^E, \pi\right\}$ is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if there exists a profile of EPIC extended direct mechanisms $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ such that for all $j \in \mathcal{J}$, (i) $\bar{\rho}_j(j) = \mu_j^E$ and (ii)*

$$V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi_j, \pi_j\right). \quad (24)$$

Proof. See Appendix A.2. ■

The next two sections show how to utilize the Revelation Principles to check if an equilibrium allocation in a specific competition model is a robust quasi ex-post equilibrium allocation that is independent of any ad hoc restrictions imposed on the model (Section 7), and to derive the set of robust quasi ex-post equilibrium allocations in our general model (Section 8).

7 Robustness in specific competition models

In most competing auctions, each seller offers a (second-price) auction and does not change his auction in response to a competing seller's deviation, that is $\bar{\rho}_k(j) = \bar{\rho}_k(j')$ for all $j, j' \in \mathcal{J}$. Further, when a seller deviates, he takes a buyer's truthful type reporting to a non-deviating seller upon selecting him as given. Therefore, an equilibrium in competing auctions is a quasi ex-post equilibrium.

In fact, most equilibria in applications in competing auctions, decentralized matching with contracts, competitive screening, directed/competitive search. etc. belong to quasi ex-post equilibria. The Strong Revelation Principle provides a way to check if an equilibrium allocation in a specific competition model in applications is a robust quasi ex-post equilibrium allocation.

The crux is that for any BIC direct mechanism μ_j available for principal j's deviation, given EPIC direct mechanisms (or dominant-strategy incentive compatible (DIC) direct mechanisms in private values) that non-deviators implements upon j's deviation, there may be multiple continuation equilibrium selection strategies that make μ_j BIC. For the robustness, we must choose the selection strategy that maximizes deviating principal j's utility given μ_j. This is the approach that Cole and Eeckhout (2003), Peters (1997), and Han (2015) all take.

Coles and Eeckhout (2003) Coles and Eeckhout (2003) consider a model of complete information with two sellers and two buyers. Each seller has one unit of an indivisible good to sell. Because buyers' valuations are identical and publicly observable, a direct mechanism is a mapping from the number of participating buyers into prices. They construct a competition model where each seller simultaneously announces a direct mechanism and then each buyer decides which seller to choose. In this model, a seller does not change his direct mechanism even if his competitor deviates. Clearly, a SPNE focused in their paper is a quasi ex-post equilibrium.[9]

One may wonder if any SNPE is a robust quasi ex-post equilibrium and hence no seller can gain in any continuation equilibrium conditional on buyers' strategies of communicating with a non-deviating seller when he deviates to any complex mechanism. Since a buyer's valuation is publicly observable, communication is degenerate in a direct mechanism and the set of all (BIC) direct mechanisms $\mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)$ available for seller j's deviation is determined independent of the other seller's direct mechanism and buyers' selection strategies.

(24) in the Strong Revelation implies that for the robustness, we only need to check if there is any continuation equilibrium selection strategy π_j which makes j's deviation to a direct mechanism profitable. Lemma 2 and Eq. (5) in Coles and Eeckhout (2003) show that an SPNE focused in their paper is based on the buyer's selection strategy that maximizes seller j's profit upon deviation to any direct mechanism, given the

[9]In the case of complete information, a direct mechanism chosen by a seller in equilibrium can be thought of as an EPIC extended direct mechanism in which the seller's price is independent of participating buyers' report on whether or not the competing seller deviates. We fix a participating buyer's truth telling to a non-deviator regardless of a deviating seller's mechanism.

competitor's direct mechanism. Therefore, invoking the Strong Revelation Principle, we can say that any SPNE in Coles and Eeckhout (2003) is a robust quasi ex-post equilibrium.[10]

Peters (1997) and Han (2015) In competing auctions in Peters (1997), J sellers face κJ buyers. The distribution of the buyer's valuation is denoted by F with its support $[0, 1]$, whereas the distribution of costs in the population of sellers is denoted by G with its support contained in $[0, 1]$.

Peters (1997) considers the finite approximation of the limit game with the infinite number of traders given the fixed ratio κ of buyers to sellers. A cutoff valuation for seller j is the infimum of the set of valuations for which buyers choose seller j with positive probability. Let H denote a distribution of cutoff valuations for the limit game.

For a finite approximation, consider the market where $J - 1$ sellers hold *second-price auctions* $\bar{\mu}_{-J} = \{\bar{\mu}_1, \ldots, \bar{\mu}_{J-1}\}$ with the distribution of the cutoff valuations $\bar{H}_J$ that converges almost everywhere to H. Let $\pi'_J(x, \mu'_J)$ be the selection probability with which a buyer with valuation x selects deviating seller J when his direct mechanism is μ'_J given $\bar{\mu}_{-J}$.

Then, for any continuation equilibrium selection strategy π'_J, the deviating seller's payoff $\hat{\Phi}_J(w, \mu'_J, H, \pi'_J)$ associated with offering a BIC direct mechanism μ'_J can be derived when w is his cost. On the other hand, let $\hat{\Phi}'_J(w, \mu'(y), H)$ denote the payoff to the deviating seller with cost w if he offers a second-price auction $\mu'(y)$ that induces a cutoff valuation y.[11]

Let $\Pi'_J(\mu'_J, H)$ be the set of all continuation equilibrium strategies π'_J of selecting the deviating seller with μ'_J given H. Let $\mathcal{M}^B_J(H)$ be the set of all BIC direct mechanisms available for each seller's deviation given H. The definition of a competitive distribution of second-price auctions in Peters (1997) can be provided as follows:

[10] A competition in direct mechanisms may not generate all possible robust quasi ex-post equilibrium allocations but still induces a large set of robust quasi ex-post equilibrium allocations in the case of complete information: The equilibrium price conditional on two participating buyers can be anything between zero and their valuation, whereas the equilibrium price conditional on a single participating buyer is exactly a half of the buyer's valuation.

[11] See Peters (1997) or Han (2015) for details on the derivations of $\hat{\Phi}_J(w, \mu'_J, H, \pi'_J)$ and $\hat{\Phi}'_J(w, \mu'(y), H)$.

Definition 5 *A competitive distribution of second-price auctions is a distribution of cutoff valuations H and a cutoff rule $y : [0, 1] \to [0, 1]$ such that for almost all w,*

1. *for all $\mu'_J \in \mathcal{M}^B_J(H)$ and all $\pi'_J \in \Pi'_J(\mu'_J, H)$*

$$\lim_{J\to\infty} \hat{\Phi}'_J(w, \mu'(y(w)), H) \geq \lim_{J\to\infty} \hat{\Phi}_J(w, \mu'_J, H, \pi'_J) \qquad (25)$$

2. *and $H(y(w)) = G(w)$.*

Theorem 5 in Peters (1997) shows that there is a competitive distribution of second-price auctions in which each seller offers a second-price auction with reserve price equal to his cost. Clearly a competitive distribution of second-price auctions is a quasi ex-post equilibrium.[12] Proposition 1 in Han (2015) shows that a competitive distribution of second-price auctions is robust in the sense that no seller can gain in any continuation equilibrium conditional on buyers' truthful valuation reporting to any non-deviators upon selecting them when he deviates to any complex mechanism. In fact, the robustness results in Peters (1999) and Han (2015) can be viewed as a corollary of the Strong Revelation Principle because (25) is a limit version of (24) in the Strong Revelation Principle for competing auctions.

8 Equilibrium characterization

We apply Revelation Principles (Theorems 1 and 2) to characterize the set of robust quasi ex-post equilibrium allocations in our general model when there are three or more agents ($I \geq 3$).[13] Because of the Weak Revelation Principle (Theorem 1), we focus on EPIC allocations that can be supported in a robust truthful quasi ex-post equilibrium without loss of generality.

For all $j \in \mathcal{J}$, fix the part of an EPIC extended direct mechanism, $(\rho^1_j, \rho^2_j) \in \Lambda^1_j \times \Lambda^2_j$ that is used when one or two agents participate. We define all possible EPIC

[12]A second-price auction chosen by a seller in equilibrium can be viewed as an EPIC extended direct mechanism where the allocation of the seller's object and monetary transfers are independent of participating buyers' reports on the identity of a deviating seller (if any). We fix a participating buyer's truth telling to a non deviator regardless of a deviating seller's mechanism.

[13] None of the results prior to Section 8 requires three or more agents in the model.

extended direct mechanisms associated with ρ_j^1 and ρ_j^2:

$$\mathbf{R}_j\left(\rho_j^1, \rho_j^2\right) \equiv \left\{\acute{\rho}_j = \left\{\acute{\rho}_j^n\right\}_{n=1}^I : \acute{\rho}_j \in \Lambda_j,\ \acute{\rho}_j^1 = \rho_j^1,\ \acute{\rho}_j^2 = \rho_j^2\right\}.$$

Given $\left(\rho_{-j}^1, \rho_{-j}^2\right)$, we also define the threshold of principal j's utility as

$$v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right) \equiv \inf_{\rho_{-j} \in \mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)} \sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_{-j}(j)\right)} V_j\left(\mu_j, \rho_{-j}(j), \phi_j, \pi_j\right),$$

where $\mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right) = \left(\mathbf{R}_k\left(\rho_k^1, \rho_k^2\right)\right)_{k \neq j}$. The threshold $v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right)$ is defined for the case where the minimum of

$$\sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_{-j}(j)\right)} V_j\left(\mu_j, \rho_{-j}(j), \phi_j, \pi_j\right)$$

over $\mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)$ does not exist. If it does, (26) below holds with weak inequality.

Theorem 3 provides the constrained characterization of robust quasi ex-post equilibrium allocations conditional on $(\rho_j^1, \rho_j^2) \in \Lambda_j^1 \times \Lambda_j^2$ for each $j \in \mathcal{J}$.

Theorem 3 (ρ^{1-2}-Characterization) *For a given $(\rho_j^1, \rho_j^2) \in \Lambda_j^1 \times \Lambda_j^2$ for each $j \in \mathcal{J}$, an EPIC allocation $\left(\mu^E, \pi\right)$ is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium with a profile of EPIC extended direct mechanisms $(\rho_1, \ldots, \rho_J)$ with $\rho_j \in \mathbf{R}_j\left(\rho_j^1, \rho_j^2\right)$ for each $j \in \mathcal{J}$ if and only if, for all $j \in \mathcal{J}$, (i) $\mu_j^E = \left\{\rho_j^n(j)\right\}_{n=1}^I$ and (ii)*

$$V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) > v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right). \tag{26}$$

Proof. For each $j \in \mathcal{J}$, we fix $\left(\rho_j^1, \rho_j^2\right) \in \Lambda_j^1 \times \Lambda_j^2$. It is clear that condition (i), $\mu_j^E = \left\{\rho_j^n(j)\right\}_{n=1}^I$, must be satisfied since $\left\{\rho_j^n(j)\right\}_{n=1}^I$ is the EPIC direct mechanism that principal j implements when no principal deviates.

Consider an EPIC allocation $\left(\mu^E, \pi\right)$ such that there exists a profile of EPIC extended direct mechanisms, $(\rho_1, \ldots, \rho_J)$ with $\rho_j \in \mathbf{R}_j\left(\rho_j^1, \rho_j^2\right)$ for each j, that satisfies condition (i): $\mu_j^E = \left\{\rho_j^n(j)\right\}_{n=1}^I$. Such a profile $\rho = (\rho_1, \ldots, \rho_J)$ must also satisfy (26) for all $j \in \mathcal{J}$ because otherwise (24) in the Strong Revelation Principle (Theorem 2) does not hold for some $j \in \mathcal{J}$. Therefore, conditions (i) and (ii) are necessary.

Given $\left(\rho_j^1, \rho_j^2\right) \in \Lambda_j^1 \times \Lambda_j^2$ that we fix for all $j \in \mathcal{J}$, now suppose that an EPIC allocation $\left(\mu^E, \pi\right)$ satisfies condition (ii), together with condition (i). This implies that for all j, we have

$$\rho_{-j}^j = \left[\rho_k^j = \left\{\rho_j^{jn}\right\}_{n=1}^I\right]_{k\neq j} \in \mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)$$

that satisfies

$$V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}^j(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_{-j}^j(j)\right)} V_j\left(\mu_j, \rho_{-j}^j(j), \phi_j, \pi_j\right). \quad (27)$$

In terms of principal k's viewpoint, ρ_k^j can be thought of as the EPIC extended direct mechanism used upon principal j's deviation.

Since $\rho_{-j}^j \in \mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)$ for all j, it is clear that $\rho_k^{j1}(j) = \rho_k^1(j)$ and $\rho_k^{j2}(j) = \rho_k^2(j)$ for all j. However, for some $n \geq 3$, we may have that $\rho_k^j \neq \rho_k^{j'}$ for some $j, j' \neq k$ with $j \neq j'$. We construct a single EPIC extended direct mechanism $\bar{\rho}_k = \{\bar{\rho}_k^n\}_{n=1}^I$ that principal k can use on and also off the paths following unilateral deviations by all the other principals as follows:

$$\bar{\rho}_k^1(j) = \rho_k^1(j), \ \forall j \in \mathcal{J} \quad (28)$$
$$\bar{\rho}_k^2(j) = \rho_k^2(j), \ \forall j \in \mathcal{J} \quad (29)$$

and for all $n \geq 3$, all $(\omega_1, \ldots, \omega_n) \in \Omega^n$,

$$\bar{\rho}_k^n\left[(j, \omega_h)_{h=1}^n\right] = \bar{\rho}_k^n\left[\left(j', \omega_i\right), (j, \omega_h)_{h\neq i}\right] = \rho_k^{jn}\left[(j, \omega_i)_{h=1}^n\right], \ \forall i, \forall j \neq k, \forall j' \quad (30)$$
$$\bar{\rho}_k^n\left[(k, \omega_h)_{h=1}^n\right] = \bar{\rho}_j^n\left[(j, \omega_i), (k, \omega_h)_{h\neq i}\right] = \mu_j^{En}(\omega_1, \ldots, \omega_n), \ \forall j. \quad (31)$$

Because $n \geq 3$, (30) and (31) show that $\bar{\rho}_k^n$ ignores the report in $\mathcal{J}$ from an agent when it is not the same as the reports from the remaining agents. Therefore, no agent has incentives to change her report in $\mathcal{J}$ from the same report submitted by the remaining agents. Since μ_j^{En} and $\rho_k^{jn}(j)$ for all $j \neq k$ are EPIC over payoff types, it implies that $\bar{\rho}_k^n$ is EPIC over $\mathcal{J} \times \Omega$ for all $n \geq 3$. Because of (28) and (29), $\bar{\rho}_k^n$ is EPIC over $\mathcal{J}$ and Ω for $n = 1, 2$. Subsequently, $\bar{\rho}_k = \{\bar{\rho}_k^n\}_{n=1}^I$ is EPIC over $\mathcal{J} \times \Omega$.

Furthermore, (28), (29), and (31) imply that $\{\bar{\rho}_k^n(k)\}_{n=1}^I = \mu_j^E$, which satisfies condition (i) in the Strong Revelation Principle (Theorem 2). On the other hand, (28), (29), and (30) imply that

$$\{\bar{\rho}_k^n(j)\}_{n=1}^I = \left\{\rho_k^{jn}(j)\right\}_{n=1}^I, \forall j \neq k. \tag{32}$$

Because $\rho_{-j}^j(j)$ satisfies (27), (32) and (27) together induce

$$V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi_j, \pi_j\right)$$

for all $j \in \mathcal{J}$. Therefore, $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ satisfies (24), i.e., condition (ii) in the Strong Revelation Principle (Theorem 2). Invoking the Strong Revelation Principle, the EPIC allocation $\left(\mu^E, \pi\right)$ is supported in a robust truthful quasi ex-post equilibrium. ■

For all $k \in \mathcal{J}$, all $j \neq k$, and all $(\rho_k^1, \rho_k^2) \in \Lambda_k^1 \times \Lambda_k^2$, we define $\mathbf{P}_k^j(\rho_k^1, \rho_k^2)$ as

$$\mathbf{P}_k^j\left(\rho_k^1, \rho_k^2\right) \equiv \left\{\mu_k^E = \left\{\mu_k^{En}\right\}_{n=1}^I : \begin{array}{c} \mu_k^{En} = \rho_k^n(j) \text{ for } n = 1, 2, \\ \mu_k^{En} \in \mathcal{D}_k^{En} \text{ for } n \geq 3 \end{array}\right\}.$$

Proposition 2 *For all $j \in \mathcal{J}$,*

$$v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right) = \inf_{\mu_{-j}^E \in \mathbf{P}_{-j}^j\left(\rho_{-j}^1, \rho_{-j}^2\right)} \sup_{\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^E\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \mu_{-j}^E\right)} V_j\left(\mu_j, \mu_{-j}^E, \phi_j, \pi_j\right), \tag{33}$$

where $\mathbf{P}_{-j}^j\left(\rho_{-j}^1, \rho_{-j}^2\right) = \left\{\mathbf{P}_k^j(\rho_k^1, \rho_k^2)\right\}_{j \neq k}$.

Proof. Every principal k is constrained to use $\acute{\rho}_k$ with $\acute{\rho}_k^n = \rho_k^n$ for $n = 1, 2$. Therefore, principal k responds principal j's deviation with $\rho_k^n(j)$ for $n = 1, 2$.

Suppose that three or more agents select principal k. We show that principal k can respond with any EPIC direct mechanism $\mu_k^{En} \in \mathcal{D}_k^{En}$ for all $n \geq 3$. To see this, note that for any $\mu_k^{En} \in \mathcal{D}_k^{En}$ for all $n \geq 3$, principal k can construct ρ_k^n such that for all $j \neq k$, all $j' \in \mathcal{J}$, all $(\omega_1, \ldots \omega_n) \in \Omega^n$ and all $i = 1, \ldots, n$

$$\rho_k^n\left[\left(j', \omega_i\right), \left(j, \omega_h\right)_{h \neq i}\right] = \mu_k^{En}\left(\omega_1, \ldots \omega_n\right) \tag{34}$$

(34) implies that when $n-1$ agents report j, one single agent's deviation to j' does not

affect principal k's action assigned by ρ_k^n. Therefore, all agents can truthfully report j to principal k when principal j deviates. It implies that all EPIC direct mechanisms μ_k^{En} in $\mathcal{D}_k^{En}$ can be implemented for $n \geq 3$. Therefore, choosing $\rho_{-j} \in \mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)$ to minimize

$$\sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_{-j}(j)\right)} V_j\left(\mu_j, \rho_{-j}(j), \phi_j, \pi_j\right)$$

is equivalent to choosing $\mu_{-j}^E \in \mathbf{P}_{-j}^j\left(\rho_{-j}^1, \rho_{-j}^2\right)$ to minimize

$$\sup_{\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^E\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \mu_{-j}^E\right)} V_j\left(\mu_j, \mu_{-j}^E, \phi_j, \pi_j\right).$$

This leads to (33). ■

Proposition 2 shows that $\mathbf{P}_{-j}^j\left(\rho_{-j}^1, \rho_{-j}^2\right)$ is the set of EPIC direct mechanisms that principals except j can implement from EPIC extended direct mechanisms in $\mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)$ upon principal j's deviation. Importantly, it shows that the threshold of principal j's robust quasi ex-post equilibrium utility $v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right)$ is expressed in terms of model primitives, i.e., principal j's BIC direct mechanisms, the other principals' EPIC direct mechanisms based on agents' continuation equilibrium selection strategy that maximizes principal j's utility. This is a key difference from the characterization in Yamashita (2010) where the threshold of principal j's (perfect Bayesian) equilibrium utility is expressed only in terms of mechanisms allowed in a competing mechanism game based on the (worst) continuation equilibrium for j.

Let $\Lambda^n = \Lambda_1^n \times \cdots \times \Lambda_J^n$ for all n. Given $(\rho^1, \rho^2) \in \Lambda^1 \times \Lambda^2$, define

$$\mathbf{E}\left(\rho^1, \rho^2\right) \equiv \left\{\left(\mu^E, \pi\right) : \begin{array}{c} \left(\mu^E, \pi\right) \text{ is an EPIC alloc. with } \mu_j^{En} = \rho_j^n(j), \ \forall j \text{ and } n = 1, 2 \\ V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) > v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right), \ \forall j \end{array}\right\}.$$

Theorem 4 (Full Characterization) *The set of robust quasi ex-post equilibrium allocations* $\mathbf{E}^*$ *is*

$$\mathbf{E}^* = \bigcup_{(\rho^1, \rho^2) \in \Lambda^1 \times \Lambda^2} \mathbf{E}\left(\rho^1, \rho^2\right).$$

Proof. $\mathbf{E}(\rho^1, \rho^2)$ is the set of EPIC allocations supportable in a robust quasi ex-post equilibrium associated with $(\rho^1, \rho^2) \in \Lambda^1 \times \Lambda^2$ by ρ^{1-2}-Characterization (Theorem 3).

Invoking the Weak Revelation Principle (Theorem 1), we conclude that $\mathbf{E}^{*}$ is the set of robust quasi ex-post equilibrium allocations. ■

8.1 Two principals

When there are only two principals, an alternative characterization of robust quasi ex-post equilibrium allocations is possible. For any $j \in \mathcal{J}$, fix his EPIC direct mechanisms $\left(\mu_j^{E1}, \mu_j^{E2}\right) \in \mathcal{D}_j^{E1} \times \mathcal{D}_j^{E2}$ for the case where one or two agents select him. Given $\left(\mu_j^{E1}, \mu_j^{E2}\right)$, define the set of EPIC extended direct mechanisms $\mathcal{R}_j\left(\mu_j^{E1}, \mu_j^{E2}\right)$ as

$$\mathcal{R}_j\left(\mu_j^{E1}, \mu_j^{E2}\right) \equiv \left\{\rho_j = \left\{\rho_j^n\right\}_{n=1}^{I} : \rho_j \in \Lambda_j,\ \rho_j^n(j) = \mu_j^{En} \text{ for } n = 1, 2\right\}.$$

and the threshold of his utility for all $j \in \mathcal{J} = \{1, 2\}$ as

$$v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right) \equiv \inf_{\rho_k \in \mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)} \sup_{\mu_j \in \mathcal{M}_j^B(\rho_k(j))} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_k(j)\right)} V_j\left(\mu_j, \rho_k(j), \phi_j, \pi_j\right),$$

where $k \neq j$. The threshold $v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right)$ is defined for the case where minimum over $\mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)$ does not exist. If it does, (35) below holds with weak inequality.

Theorem 5 (μ^{E1-2}-Characterization) *An EPIC allocation $\left(\mu^E, \pi\right)$ is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if, for all $j \in \mathcal{J}$*

$$V_j\left(\mu_j^E, \mu_k^E, \phi_j, \pi_j\right) > v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right), \tag{35}$$

where $k \neq j$ and $\mu_k^{E1}$and μ_k^{E2} in v_j^* are principal k's EPIC direct mechanisms for $n = 1, 2$ in $\mu_k^E = \left\{\mu_k^{En}\right\}_{n=1}^{I}$.

Proof. See Appendix A.3. ■

μ^{E1-2}-Characterization (Theorem 5) is less constrained than ρ^{1-2}-Characterization (Theorem 3) in that it is conditional only on the EPIC direct mechanisms that are implemented on the equilibrium path when one or two agents select a principal. However, μ^{E1-2}-Characterization cannot be extended to the case with three or more principals.

For example, consider a model with three principals. Suppose that given an EPIC allocation $\left(\mu^E, \pi\right)$, (35) is satisfied for all $j \in \mathcal{J}$. This means that for all j, we have that $\rho_{-j}^{j} \in \mathcal{R}_{-j}\left(\mu_{-j}^{E1}, \mu_{-j}^{E2}\right)$ such that

$$V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}^{j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_{-j}^{j}(j)\right)} V_j\left(\mu_j, \rho_{-j}^{j}(j), \phi_j, \pi_j\right).$$

For principal k, ρ_k^j and $\rho_k^{j'}$ may be different in that for some n

$$\rho_k^{jn}(j') \neq \rho_k^{j'n}(j') \text{ and } \rho_k^{j'n}(j) \neq \rho_k^{jn}(j) \text{ for } j, j' \neq k \text{ and } j \neq j'. \tag{36}$$

Fix n satisfying (36), which implies that neither ρ_k^{jn} nor $\rho_k^{j'n}$ can be a single EPIC extended direct mechanism given n participating agents that principal k uses for both unilateral deviations by principals j and j'. This problem does not arise in a model with two principals because there is only one potential deviating principal.

The question is whether we can find a single EPIC extended direct mechanism $\bar{\rho}_k^n$ for all n such that $\bar{\rho}_k^n(j) = \rho_k^{jn}(j)$ and $\bar{\rho}_k^n(j') = \rho_k^{j'n}(j')$. It is not the problem to construct such $\bar{\rho}_k^n$ for $n \geq 3$. It is also not the problem for $n = 1$. Why? When only one agent selects principal k, the EPIC property of ρ_k^j and $\rho_k^{j'}$ implies that

$$u_k^1\left(\rho_k^{j1}(\ell, \omega_1), \omega_1\right) = u_k^1\left(\mu_k^{E1}(\omega_1), \omega_1\right) = u_k^1\left(\rho_k^{j'1}(\ell, \omega_1), \omega_1\right) \ \forall (\ell, \omega_1) \in \mathcal{J} \times \Omega,$$
$$u_k^1\left(\rho_k^{j1}(\ell, \omega_1), \omega_1\right) \geq u_k^1\left(\rho_k^{j1}(\ell, \omega_1'), \omega_1\right) \ \forall \ell \in \mathcal{J}, \forall \omega_1', \omega_1 \in \Omega,$$
$$u_k^1\left(\rho_k^{j'1}(\ell, \omega_1), \omega_1\right) \geq u_k^1\left(\rho_k^{j'1}(\ell, \omega_1'), \omega_1\right) \ \forall \ell \in \mathcal{J}, \forall \omega_1', \omega_1 \in \Omega.$$

Therefore, we can simply choose $\bar{\rho}_k^1$ satisfying that for all $\omega_1 \in \Omega$, (i) $\bar{\rho}_k^1(j, \omega_1) = \rho_k^{j1}(j, \omega_1)$, (ii) $\bar{\rho}_k^1(j', \omega_1) = \rho_k^{j'1}(j', \omega_1)$, and (iii) $\bar{\rho}_k^1(k, \omega_1) = \mu_k^{E1}(\omega_1)$. Given $\bar{\rho}_k^1$, the single participating agent will truthfully report her true type and the identity of a deviating principal (if any).

However, it is the problem for $n = 2$. Principal k wants to implement $\rho_k^{j2}(j)$ when principal j deviates but $\rho_k^{j'2}(j')$ when principal j' deviates. Because of (36) with $n = 2$, neither ρ_k^{j2} nor $\rho_k^{j'2}$ can be a single EPIC extended direct mechanism given two participating agents that principal k uses for both unilateral deviations by principals j and j'. It is also generally not possible to construct an alternative single EPIC extended direct mechanism given two participating agents $\bar{\rho}_k^2$ such that $\bar{\rho}_k^2(j) = \rho_k^{j2}(j)$ and $\bar{\rho}_k^2(j') = \rho_k^{j'2}(j')$. The reason is that when one agent reports j

and the other j', principal k does not know which agent tells the true identity of the deviating principal. Such a problem with $n = 2$ is absent in ρ^{1-2}-Characterization because it fixes ρ_k^n for $n = 1, 2$ and all k.[14]

For all $k \in \mathcal{J}$, we define $\mathcal{P}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)$ as

$$\mathcal{P}_k\left(\mu_k^{E1}, \mu_k^{E2}\right) \equiv$$
$$\left\{ \acute{\mu}_k^E = \left\{\acute{\mu}_k^{En}\right\}_{n=1}^I : \begin{array}{c} \acute{\mu}_k^{En} = \acute{\rho}_k^n(j) \text{ for } j \neq k,\ n = 1, 2, \text{ and some } \acute{\rho}_k \in \mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right), \\ \acute{\mu}_k^{En} \in \mathcal{D}_k^{En} \text{ for } n \geq 3 \end{array} \right\}.$$

Proposition 3 *For all $j \in \mathcal{J}$,*

$$v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right) = \inf_{\acute{\mu}_k^E \in \mathcal{P}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)} \sup_{\mu_j \in \mathcal{M}_j^B\left(\acute{\mu}_k^E\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \acute{\mu}_k^E\right)} V_j\left(\mu_j, \acute{\mu}_k^E, \phi_j, \pi_j\right). \quad (37)$$

Proof. Principal k chooses an EPIC extended direct mechanism in $\mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)$ to minimize

$$\sup_{\mu_j \in \mathcal{M}_j^B(\rho_k(j))} \sup_{\pi_j \in \Pi_j\left(\mu_j, \rho_k(j)\right)} V_j\left(\mu_j, \rho_k(j), \phi_j, \pi_j\right)$$

given $\left(\mu_k^{E1}, \mu_k^{E2}\right) \in \mathcal{D}_k^{E1} \times \mathcal{D}_k^{E2}$. This implies that when one or two agents select principal k upon principal j's deviation, the EPIC direct mechanisms he can use are $\acute{\rho}_k^1(j)$ and $\acute{\rho}_k^2(j)$ for any $\acute{\rho}_k \in \mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)$. On the other hand, analogously to the proof of Proposition 2, we can show that when three or more agents select him, he can use any EPIC direct mechanisms upon principal j's deviation. It is then straightforward to show (37). ■

Define

$$\Xi\left(\mu_1^{E1}, \mu_1^{E2}, \mu_2^{E1}, \mu_2^{E2}\right) \equiv$$
$$\left\{ \left(\acute{\mu}^E, \pi\right) : \begin{array}{c} \left(\acute{\mu}^E, \pi\right) \text{ is EPIC with } \acute{\mu}_j^{En} = \mu_j^{En},\ \forall j \text{ and } n = 1, 2 \\ V_j\left(\acute{\mu}_j^E, \acute{\mu}_k^E, \phi_j, \pi_j\right) > v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right),\ \forall j \end{array} \right\}$$

[14] Alternative to ρ^{1-2}- Characterization (Theorem 3) for a model with any multiple number of principals, μ^{E1}-ρ^2 Characterization can be established. This characterization derives the set of robust quasi ex-post equilibrium allocations supportable with the class of EPIC extended direct mechanisms in which each principal k uses an EPIC extended direct mechanism that implements μ_k^{E1} conditional on only one agent's participation on the equilibrium path and ρ_k^2 is the EPIC extended direct mechanism conditional on two agents' participation on or off the path.

Theorem 6 (Full Characterization) *The set of robust quasi ex-post equilibrium allocations is*

$$\mathbf{E}^{*} = \bigcup_{\left(\mu_1^{E1},\mu_1^{E2},\mu_2^{E1},\mu_2^{E2}\right)\in\mathcal{D}_1^{E1}\times\mathcal{D}_1^{E2}\times\mathcal{D}_2^{E1}\times\mathcal{D}_2^{E2}} \Xi\left(\mu_1^{E1},\mu_1^{E2},\mu_2^{E1},\mu_2^{E2}\right).$$

We can prove Theorem 6 analogously to the proof of Theorem 4, so it is omitted.

8.2 Agent's preference restrictions for $n = 2$

Principal k can use any EPIC direct mechanism $\mu_k^{En} \in \mathcal{D}_k^{En}$ to respond a deviation by any of his competing principals when three or more agents select k ($n \geq 3$) It is also possible in the case of only two participating agents ($n = 2$) if the preference relation satisfies Assumption 1 below is satisfied.

First, we define lower contour sets. For all $k \in \mathcal{J}$, all $\alpha, \alpha' \in \mathcal{A}_k$ and all $\omega_1, \omega_2, \omega_1', \omega_2' \in \Omega$, define lower contour sets as follows

$$\begin{aligned}
\mathcal{L}_k\left(\alpha,\omega_2,\omega_1'\right) &\equiv \left\{\alpha'' \in \mathcal{A}_k : u\left(\alpha'',(\omega_2,\omega_1')\right) \leq u\left(\alpha,(\omega_2,\omega_1')\right)\right\} \\
\mathcal{L}_k\left(\alpha',\omega_1,\omega_2'\right) &\equiv \left\{\alpha'' \in \mathcal{A}_k : u\left(\alpha'',(\omega_1,\omega_2')\right) \leq u\left(\alpha',(\omega_1,\omega_2')\right)\right\}
\end{aligned}$$

When ω_2 is agent 2's type and α is the action of principal k who are chosen by only agents 1 and 2, the utility of agent 1 of type ω_1' is $u\left(\alpha,(\omega_2,\omega_1')\right)$. Given (ω_2,ω_1'), we can identify the set of principal k's actions agent 1 of type ω_1' weakly less prefers to α. This is the lower contour set $\mathcal{L}_k\left(\alpha,\omega_2,\omega_1'\right)$ for agent 1, defined above. We can analogously define the lower contour set $\mathcal{L}_k\left(\alpha',\omega_1,\omega_2'\right)$ for agent 2.

Assumption 1 *For all $k \in \mathcal{J}$, all $\alpha, \alpha' \in \mathcal{A}_k$, and all $\omega_1, \omega_2 \in \Omega$*

$$\left[\bigcap_{\omega_1'\in\Omega} \mathcal{L}_k\left(\alpha,\omega_2,\omega_1'\right)\right] \cap \left[\bigcap_{\omega_2'\in\Omega} \mathcal{L}_k\left(\alpha',\omega_1,\omega_2'\right)\right] \neq \varnothing \tag{38}$$

Consider agents 1 and 2 such that (i) agent 1 is in a situation where agent 2's type is ω_2 and the action of principal k they select is α and (ii) agent 2 is in a situation where agent 1's type is ω_1 and the action of principal k they select is α'. Assumption 1 says that conditional on (ω_2,α) and (ω_1,α'), there is principal k's action that agent

1 weakly less prefers to α and agent 2 weakly less prefers to α' regardless of their types respectively.

Lemma 4 *If Assumption 1 is satisfied, any EPIC direct mechanism* $\mu_k^{E2} \in \mathcal{D}_k^{E2}$ *can be assigned to* $\rho_k^2(j)$ *for all* $j \in \mathcal{J}$ *for an EPIC extended direct mechanism* $\rho_k = \{\rho_k^n\}_{n=1}^I$.

Proof. Consider an EPIC extended direct mechanism $\rho_k = \{\rho_k^n\}_{n=1}^I$. Take an arbitrary EPIC direct mechanism in $\mathcal{D}_k^{E2}$ for $\rho_k^2(j)$ for each $j \in \mathcal{J}$.

Fix any pair of type reports $(\omega_1, \omega_2) \in \Omega^2$. Let's call one of the two agents who select principal k agent 1 and the other agent 2. Suppose that agents 1 and 2 report (j, ω_1) and (j', ω_2) to principal k respectively. Because of Assumption 1, there exists $\tilde{\alpha}(\omega_1, \omega_2) \in \mathcal{A}_k$ such that

$$\tilde{\alpha}(\omega_1, \omega_2) \in \left[\bigcap_{\omega_1' \in \Omega} \mathcal{L}_k\left(\rho_k^2\left[(j', \omega_1), (j', \omega_2)\right], \omega_2, \omega_1'\right)\right] \cap \left[\bigcap_{\omega_2' \in \Omega} \mathcal{L}_k\left(\rho_k^2\left[(j, \omega_1), (j, \omega_2)\right], \omega_1, \omega_2'\right)\right] \quad (39)$$

We then assign $\tilde{\alpha}(\omega_1, \omega_2)$ for $\rho_k^2\left[(j, \omega_1), (j', \omega_2)\right]$:

$$\rho_k^2\left[(j, \omega_1), (j', \omega_2)\right] = \tilde{\alpha}(\omega_1, \omega_2) \quad (40)$$

(39) and (40) imply that

$$u\left(\rho_k^2\left[(j, \omega_1), (j', \omega_2)\right], (\omega_2, \omega_1')\right) \leq u\left(\rho_k^2\left[(j', \omega_1), (j', \omega_2)\right], (\omega_2, \omega_1')\right), \ \forall \omega_1' \in \Omega, \quad (41)$$

$$u\left(\rho_k^2\left[(j, \omega_1), (j', \omega_2)\right], (\omega_1, \omega_2')\right) \leq u\left(\rho_k^2\left[(j, \omega_1), (j, \omega_2)\right], (\omega_1, \omega_2')\right), \ \forall \omega_2' \in \Omega. \quad (42)$$

On the other hand, $\rho_k^2(j)$ and $\rho_k^2(j')$ are EPIC direct mechanisms and hence,

$$u\left(\rho_k^2\left[(j', \omega_1), (j', \omega_2)\right], (\omega_2, \omega_1')\right) \leq u\left(\rho_k^2\left[(j', \omega_1'), (j', \omega_2)\right], (\omega_2, \omega_1')\right), \ \forall \omega_1' \in \Omega, \quad (43)$$

$$u\left(\rho_k^2\left[(j, \omega_1), (j, \omega_2)\right], (\omega_1, \omega_2')\right) \leq u\left(\rho_k^2\left[(j, \omega_1), (j, \omega_2')\right], (\omega_1, \omega_2')\right), \ \forall \omega_2' \in \Omega. \quad (44)$$

(41) and (43), and (42) and (44) respectively lead to

$$
\begin{aligned}
u\left(\rho_k^2\left[(j,\omega_1),(j',\omega_2)\right],(\omega_2,\omega_1')\right) &\leq u\left(\rho_k^2\left[(j',\omega_1'),(j',\omega_2)\right],(\omega_2,\omega_1')\right),\ \forall\omega_1'\in\Omega,\\
u\left(\rho_k^2\left[(j,\omega_1),(j',\omega_2)\right],(\omega_1,\omega_2')\right) &\leq u\left(\rho_k^2\left[(j,\omega_1),(j,\omega_2')\right],(\omega_1,\omega_2')\right),\ \forall\omega_2'\in\Omega.
\end{aligned}
$$

The inequality above show that it is ex-post optimal for each agent to report her true type and the message in $\mathcal{J}$ that is reported by the other agent. ■

With slight abuse of notation, let

$$
\mathcal{P}_k^j\left(\mu_k^{E1}\right)\equiv\left\{\acute{\mu}_k^E=\left\{\acute{\mu}_k^{En}\right\}_{n=1}^I:\begin{array}{c}\acute{\mu}_k^{E1}=\rho_k^1(j)\text{ for some }\rho_k\in\Lambda_k\text{ with }\rho_k^1(k)=\mu_k^{E1}\\ \acute{\mu}_k^{En}\in\mathcal{D}_k^{En}\text{ for }n\geq 2.\end{array}\right\}
$$

for all $k\in\mathcal{J}$ and all $j\neq k$.

Theorem 7 (μ^{E1}-Characterization) *An EPIC allocation $\left(\mu^E,\pi\right)$ is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if, for all $j\in\mathcal{J}$*

$$
V_j\left(\mu_j^E,\mu_{-j}^E,\phi_j,\pi_j\right)>v_j^*\left(\mu_{-j}^{E1}\right), \tag{45}
$$

where $\mu_{-j}^{E1}=\left\{\mu_k^{E1}\right\}_{k\neq j}$ in v_j^* are EPIC direct mechanisms for all principals $k\neq j$ for $n=1$ in $\mu_k^E=\left\{\mu_k^{En}\right\}_{n=1}^I$ and

$$
v_j^*\left(\mu_{-j}^{E1}\right)\equiv\inf_{\acute{\mu}_{-j}^E\in\mathcal{P}_{-j}\left(\mu_{-j}^{E1}\right)}\ \sup_{\mu_j\in\mathcal{M}_j^B\left(\acute{\mu}_{-j}^E\right)}\ \sup_{\pi_j\in\Pi_j\left(\mu_j,\acute{\mu}_{-j}^E\right)} V_j\left(\mu_j,\acute{\mu}_{-j}^E,\phi_j,\pi_j\right). \tag{46}
$$

Proof. Given Lemma 4, the proof can be done analogously to the proofs of Theorem 4 and Proposition 2. In particular, because of Lemma 4, principal $k\neq j$ can implement any EPIC direct mechanism $\acute{\mu}_k^{En}\in\mathcal{D}_k^{En}$ for $n\geq 2$ upon principal j's deviation. When $n=1$, he is restricted to implement $\rho_k^1(j)$ upon principal j's deviation for some $\rho_k\in\Lambda_k$ with $\rho_k^1(k)=\mu_k^{E1}$. Therefore, $\mathcal{P}_k^j\left(\mu_k^{E1}\right)$ is the set of EPIC direct mechanisms that principal k can implement upon principal j's deviation. ■

For $\mu^{E1}=\left(\mu_1^{E1},\ldots,\mu_J^{E1}\right)\in\mathcal{D}_1^{E1}\times\cdots\times\mathcal{D}_J^{E1}$, define

$$
\Phi\left(\mu^{E1}\right)\equiv\left\{\left(\acute{\mu}^E,\pi\right):\begin{array}{c}\left(\acute{\mu}^E,\pi\right)\text{ is EPIC with }\acute{\mu}_j^{E1}=\mu_j^{E1},\ \forall j\\ V_j\left(\acute{\mu}_j^E,\acute{\mu}_{-j}^E,\phi_j,\pi_j\right)>v_j^*\left(\mu_{-j}^{E1}\right),\ \forall j\end{array}\right\}
$$

Theorem 8 (Full Characterization) *The set of robust quasi ex-post equilibrium allocations is*

$$\mathbf{E}^{*}=\bigcup_{\mu^{E1}\in\mathcal{D}_{1}^{E1}\times\cdots\times\mathcal{D}_{J}^{E1}}\Phi\left(\mu^{E1}\right).$$

The proof of Theorem 8 is straightforward, so it is omitted.

9 Conclusion

Equilibrium analysis in a competing mechanism game with a restricted set of mechanisms potentially faces two issues, given a solution concept adopted for it. One is that it might generate a spurious equilibrium allocation that would disappear once principals are allowed to use more complex mechanisms. The other is that it may not generate equilibrium allocations that could have been generated in a competing mechanism game without restrictions on mechanisms.

It is challenging to address these two issues because of the complexity of communication on market information. Our paper formulates the solution concept of quasi ex-post equilibrium and refines it with the robustness to deal with multiplicity of continuation equilibria. We establish two revelation principles for robust quasi ex-post equilibria. It is shown that we only need to focus on the set of EPIC extended direct mechanisms on the path and the set of direct mechanisms, off the path for a deviating principal, which are BIC conditional on EPIC direct mechanisms implemented by non-deviators upon his deviation.

Because the set of EPIC extended direct mechanisms includes various classes of mechanisms employed in applications, the Strong Revelation Principle allows us to check if an equilibrium allocation in a specific competition model in applications is a robust quasi ex-post equilibrium allocation. The Weak Revelation Principle, in tandem with the Strong Revelation Principle, leads to the characterization of the set of robust quasi ex-post equilibrium allocations independent of any ad hoc feature in mechanisms adopted in a specific model of competition.

Because the Revelation Principles established in our paper are tractable, we hope that they can be applied for robust equilibrium analysis of competition among multiple principals in various applications. One interesting but less studied problem is competing provision of excludable public goods despite its wide applicability such as competing excludable platforms (e.g., Android vs. Apple), competing internet services, competing congestible public goods, etc. Equilibrium analysis in a general model of competition for this problem will enhance our understanding of the impacts and consequences of decentralization of excludable public good provisions through market competition. We leave this for future research.

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A Appendix

A.1 Action Externalities among principals

In our model, principal j's utility function is $v_j^n : \mathcal{A}_j \times \Omega^I \to \mathbb{R}$ when n agents select him. Therefore, there are no action externalities among principals. While this set up is consistent with most applications, the logic and intuition behind our results do not depend on it.

Therefore, our results go through with proper modifications even when we allow for action externalities among principals. Let us start with formulating each principal j's utility when there are action externalities among principals. $\mathcal{A}_j$ for each $j \in \mathcal{J}$ is assumed to be sufficient general to include a contingent action, which specifies an action conditional on the set of agents who select principal j. Therefore, we can set up principal j's utility function simply as $v_j : \mathcal{A} \times \Omega^I \to \mathbb{R}$.

Each γ_j^n is a mapping from $(M_j)^n$ into $\mathcal{A}_j$. Principal j's mechanism is a finite sequence of such mappings from $n = 1$ to $n = J$, i.e., $\gamma_j = \left\{\gamma_j^n\right\}_{n=1}^I \in \Gamma_j$. For any γ_j, there is a strategically equivalent single mechanism $\gamma_j^* : \left(M_j^*\right)^I \to \mathcal{A}_j$, where $M_j^* \equiv M_j \cup \{m^*\}$ is the message space for each agent in γ_j^*. Principal j takes m^* for an agent if and only if she does not select principal j. An agent sends a message in M_j if and only if she selects principal j. For any $\gamma_j = \left\{\gamma_j^n\right\}_{n=1}^I \in \Gamma_j$, define $\gamma_j^* : \left(M_j^*\right)^I \to \mathcal{A}_j$ as, for all $n \in \{1, \ldots, I\}$ and all $(m_{1j}, \ldots, m_{nj}) \in (M_j)^n$

$$\gamma_j^* \left((m_{1j}, \ldots, m_{nj}), (m^*)^{n-1} \right) = \gamma_j^n (m_{1j}, \ldots, m_{nj}) \tag{47}$$

Let Γ_j^* be the set of mechanisms for principal j that can be obtained through (47) for all mechanisms in Γ_j. Let $\Gamma^* \equiv \Gamma_1^* \times \cdots \times \Gamma_J^*$. $p_{ij} \in \{0, 1\}$ denote agent i's participation decision for principal j's mechanism. $p_{ij} = 1$ means that agent i selects principal j and $p_{ij} = 0$ means that agent i does not select principal j. A valid participation decision is $\sum_{j=1}^J p_{ij} \leq 1$ for all $i \in \{1, \ldots, I\}$.

Fix $c_j : \Gamma \times \Omega \to M_j$ for all $j \in \mathcal{J}$. For all $(\gamma, \omega_i) \in \Gamma \times \Omega$, define $c_j^* (\gamma^*, \omega_i, p_{ij})$

such that

$$c_j^*\left(\gamma^*,\omega_i,p_{ij}\right)=\begin{cases} c_j\left(\gamma,\omega_i\right) & \text{if } p_{ij}=1 \\ m^* & \text{otherwise} \end{cases}, \tag{48}$$

where $\gamma^*=(\gamma_1^*,\ldots,\gamma_J^*)$ is the profile of mechanisms in Γ^* with each γ_j^* derived from γ_j in $\gamma=(\gamma_1,\ldots,\gamma_J)$ according to (47). Let $s_0=[0,\ldots,0]$ be the null vector of dimension J. Let $s_k=[0,\ldots,0,1,0,\ldots,0]$ be the vector of dimension J, where only the kth component is 1 for $k=1,\ldots,J$. Let $S\equiv\{s_0,s_1,\ldots,s_J\}$. With slight abuse of notation, for all $j\in\mathcal{J}$ and all $k=0,1,\ldots,J$, let

$$p_{ij}\left(s_k\right)=\begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise} \end{cases} \tag{49}$$

Fix $\left(\pi_j:\Gamma\times\Omega\to[0,1]\right)_{j=1}^{J}$. For all $(\gamma,\omega_i)\in\Gamma\times\Omega$, define $\pi^*\left(\gamma^*,\omega_i\right)$ such that

$$\pi^*\left(\gamma^*,\omega_i\right)\left(s_k\right)=\begin{cases} \pi_k\left(\gamma,\omega_i\right) & \text{if } k\in\mathcal{J} \\ 1-\sum_{j=1}^{J}\pi_j\left(\gamma,\omega_i\right) & \text{otherwise} \end{cases} \tag{50}$$

Fix $\{c,\pi\}=\{(c_1,\ldots,c_J),(\pi_1,\ldots,\pi_J)\}$. Suppose that principal j announces γ_j given a profile of other principals' mechanisms γ_{-j}. Given (47), (48), (49), and (50), principal j's ex-ante utility becomes

$$\begin{aligned} V_j\left(\gamma_j,\gamma_{-j},c,\pi\right)\equiv\int_\Omega\cdots\int_\Omega\times\sum_{s_k^1\in S}\times\ldots\times\sum_{s_k^J\in S} \\ v_j\left(\gamma_1^*\left(\left[c_1^*\left(\gamma^*,\omega_i,p_{i1}\left(s_k^i\right)\right)\right]_{i=1}^{I}\right),\ldots,\gamma_J^*\left(\left[c_J^*\left(\gamma^*,\omega_i,p_{iJ}\left(s_k^i\right)\right)\right]_{i=1}^{I}\right),(\omega_1,\ldots,\omega_I)\right) \\ \times\pi^*\left(\gamma^*,\omega_1\right)\left(s_k^1\right)\times\cdots\times\pi^*\left(\gamma^*,\omega_I\right)\left(s_k^I\right) \\ \times dF(\omega_1)\times\cdots\times dF(\omega_I) \end{aligned} \tag{51}$$

Compared to the ex-ante utility $V_j\left(\gamma_j,\gamma_{-j},c_j,\pi_j\right)$ without action externalities among all principals, principal j's ex-ante utility $V_j\left(\gamma_j,\gamma_{-j},c,\pi\right)$ also depends on agents' strategies of selecting and communicating with any other principal as well. This requires that we modify the results in our paper accordingly.

Definitions of a (perfect Bayesian) equilibrium and a quasi ex-post equilibrium can be modified accordingly. The definition of the robustness of a quasi ex-post equilibrium is modified as follows.

Definition 6 *A quasi ex-post equilibrium* $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ *is* robust *if for all* $j \in \mathcal{J}$, *all* $\gamma_j, \gamma'_j \neq \tilde{\gamma}_j$

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}, \tilde{\pi}) \geq V_j(\gamma'_j, \tilde{\gamma}_{-j}, c', \pi')$$

for any continuation equilibrium (c', π') *at* $\left(\gamma'_j, \tilde{\gamma}_{-j}\right)$ *with* c'_{-j} *satisfying (6) for all* $k \neq j$ *given* γ_j.

Let $\Pi\left(\mu_j, \mu^E_{-j}\right)$ be the set of all continuation equilibrium strategies $(\pi_1, \ldots, \pi_J)$ of selecting one of principals conditional on truthful type reporting to every principal upon selecting them when μ^E_{-j} are the mechanisms of principals except j and $\mu_j \in \mathcal{M}^B_j\left(\mu^E_{-j}\right)$ is principal j's mechanism. Proposition 1 is modified as follows.

Proposition 4 *A quasi ex-post equilibrium* $\{\tilde{\gamma}, \tilde{c}, \tilde{\pi}\}$ *is robust if and only if*

$$V_j(\tilde{\gamma}_j, \tilde{\gamma}_{-j}, \tilde{c}, \tilde{\pi}) \geq \sup_{\mu^E_{-j} \in \tilde{G}^j_{-j}} \sup_{\mu_j \in \mathcal{M}^B_j\left(\mu^E_{-j}\right)} \sup_{\pi \in \Pi\left(\mu_j, \mu^E_{-j}\right)} V_j(\mu_j, \mu^E_{-j}, \phi, \pi). \tag{52}$$

Corollary 2 is modified as follows.

Corollary 3 *A truthful quasi ex-post equilibrium* $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ *is robust if and only if, for all* $j \in \mathcal{J}$

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}, \bar{\pi}\right) \geq \sup_{\mu_j \in \mathcal{M}^B_j\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi \in \Pi\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi, \pi\right), \tag{53}$$

where $\bar{\rho}_{-j}(j) = (\bar{\rho}_k(j))_{k \neq j}$.

The statement of the Weak Revelation Principle requires no modification, whereas the statement of the Strong Revelation Principle is modified as follows.

Theorem 9 (Strong Revelation Principle) *Any EPIC allocation* $\left\{\mu^E, \pi\right\}$ *is supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium if and only if there exists a profile of EPIC extended direct mechanisms* $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ *such that for all* $j \in \mathcal{J}$, *(i)* $\bar{\rho}_j(j) = \mu^E_j$ *and (ii)*

$$V_j\left(\mu^E_j, \mu^E_{-j}, \phi, \pi\right) \geq \sup_{\mu_j \in \mathcal{M}^B_j\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi \in \Pi\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi, \pi\right). \tag{54}$$

For equilibrium characterization, the threshold of principal j's utility conditional on $\left(\rho_{-j}^1, \rho_{-j}^2\right)$ is modified as

$$v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right) \equiv \inf_{\rho_{-j} \in \mathbf{R}_{-j}\left(\rho_{-j}^1, \rho_{-j}^2\right)} \sup_{\mu_j \in \mathcal{M}_j^B\left(\rho_{-j}(j)\right)} \sup_{\pi \in \Pi\left(\mu_j, \rho_{-j}(j)\right)} V_j\left(\mu_j, \rho_{-j}(j), \phi, \pi\right).$$

ρ^{1-2}-Characterization can be established given each principal j's utility, (51). Proposition 2 is modified as follows.

Proposition 5 *For all $j \in \mathcal{J}$,*

$$v_j^*\left(\rho_{-j}^1, \rho_{-j}^2\right) = \inf_{\mu_{-j}^E \in \mathbf{P}_{-j}^j\left(\rho_{-j}^1, \rho_{-j}^2\right)} \sup_{\mu_j \in \mathcal{M}_j^B\left(\mu_{-j}^E\right)} \sup_{\pi \in \Pi\left(\mu_j, \mu_{-j}^E\right)} V_j\left(\mu_j, \mu_{-j}^E, \phi, \pi\right). \tag{55}$$

The characterization results for the cases with two principals or the agent's preference restrictions with $n = 2$ can be modified accordingly as well.

A.2 Proof of Theorem 2

Fix an EPIC allocation $\left\{\mu^E, \pi\right\}$. Suppose that there exists a profile of EPIC extended direct mechanisms $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ that satisfies conditions (i) and (ii) for all $j \in \mathcal{J}$. Construct a robust truthful quasi ex-post equilibrium $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ as follows. On the path following no principal's deviation, agents follow $\{\bar{c}, \bar{\pi}\}$ such that for all $j \in \mathcal{J}$, all $i \in \{1, \ldots, I\}$, and all $\omega_i \in \Omega$,

$$\bar{c}_j\left(\bar{\rho}, \omega_i\right) = \left(j, \omega_i\right) \tag{56}$$

$$\bar{\pi}_j\left(\bar{\rho}, \omega_i\right) = \pi_j\left(\mu^E, \omega_i\right) \tag{57}$$

$\{\bar{c}, \bar{\pi}\}$ satisfying (56) and (57) leads to a continuation equilibrium at $\bar{\rho}$ because $\left\{\mu^E, \pi\right\}$ is an EPIC allocation. Further, we have that

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) = V_j\left(\mu_j^E, \mu_{-j}^E, \phi_j, \pi_j\right). \tag{58}$$

Off the path following principal j's deviation to any $\gamma_j \neq \bar{\rho}_j$, we fix agents' communication strategy for each non-deviating principal k as, for all $k \neq j$, all $i \in \{1, \ldots, I\}$, and all $\omega_i \in \Omega$,

$$\bar{c}_k\left(\gamma_j, \bar{\rho}_{-j}, \omega_i\right) = (j, \omega_i). \tag{59}$$

Given $\bar{c}_k$ specified in (59), we pick $\bar{c}_j\left(\gamma_j, \bar{\rho}_{-j}, \cdot\right)$ and $\bar{\pi}\left(\gamma_j, \bar{\rho}_{-j}, \cdot\right)$ such that $\{\bar{c}_j, \bar{c}_{-j}, \bar{\pi}\}$ leads to a continuation equilibrium at $\left(\gamma_j, \bar{\rho}_{-j}\right)$.

Because $\bar{c}_k$ specified in (59) is truth telling, non-deviating principals implement $\bar{\rho}_{-j}(j)$ for all $\gamma_j \neq \bar{\rho}_j$. Further, γ_j and $\bar{c}_j$ induces a BIC direct mechanism $\bar{g}_j^n\left(\gamma_j\right) \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)$ such that for all $n \in \{1, \ldots, I\}$ and all $(\omega_1, \ldots, \omega_n) \in \Omega^n$

$$\bar{g}_j^n\left(\gamma_j\right)(\omega_1, \ldots, \omega_n) \equiv \gamma_j^n\left[\bar{c}_j(\gamma_j, \bar{\rho}_{-j}, \omega_1), \ldots, \bar{c}_j(\gamma_j, \bar{\rho}_{-j}, \omega_n)\right].$$

Because $\bar{g}_j^n\left(\gamma_j\right) \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)$, we have that

$$\sup_{\mu_j \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi_j, \pi_j\right) \geq V_j\left(\gamma_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) \tag{60}$$

(24), (58), and (60) imply that for all $j \in \mathcal{J}$

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) \geq V_j\left(\gamma_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right), \tag{61}$$

$$V_j\left(\bar{\rho}_j, \bar{\rho}_{-j}, \bar{c}_j, \bar{\pi}_j\right) \geq \sup_{\mu_j \in \mathcal{M}_j^B\left(\bar{\rho}_{-j}(j)\right)} \sup_{\pi_j \in \Pi_j\left(\mu_j, \bar{\rho}_{-j}(j)\right)} V_j\left(\mu_j, \bar{\rho}_{-j}(j), \phi_j, \pi_j\right). \tag{62}$$

(61) shows that $\{\bar{\rho}, \bar{c}, \bar{\pi}\}$ is a truthful quasi ex-post equilibrium and (62) shows that it is robust due to Corollary 2.

Suppose that given an EPIC allocation $\left\{\mu^E, \pi\right\}$,there is no profile of EPIC extended direct mechanisms $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ that satisfies both conditions (i) and (ii) for all $j \in \mathcal{J}$. This implies that for every profile of EPIC extended direct mechanisms $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$, condition (i) is not satisfied for some j or condition (ii) is not satisfied for some j. If condition (i) is not satisfied for some j, $\left\{\mu^E, \pi\right\}$ cannot be supported on the equilibrium. If condition (ii) is not satisfied for some j, the robustness is violated. Therefore, an EPIC allocation $\left\{\mu^E, \pi\right\}$ can be supported as an equilibrium allocation in a robust truthful quasi ex-post equilibrium only if there exists a profile of EPIC extended direct mechanisms $\bar{\rho} = (\bar{\rho}_1, \ldots, \bar{\rho}_J)$ that satisfies both conditions (i) and (ii) for all $j \in \mathcal{J}$.

A.3 Proof of Theorem 5

Suppose that (35) is not satisfied: For some $j \in \mathcal{J}$

$$V_j\left(\mu_j^E, \mu_k^E, \phi_j, \pi_j\right) < v_j^*\left(\mu_k^{E1}, \mu_k^{E2}\right).$$

Then, there does not exist $\rho_k \in \mathcal{R}_k\left(\mu_k^{E1}, \mu_k^{E2}\right)$ such that (24), i.e., condition (ii) in the Strong Revelation Principle (Theorem 2) holds under ρ_k. Therefore, any EPIC allocation $\left(\mu^E, \pi\right)$ is supported in a robust quasi ex-post equilibrium only if (35) is satisfied.

Suppose that (35) is satisfied for an EPIC allocation $\left(\mu^E, \pi\right)$. Then, there exists $\rho = (\rho_1, \rho_2) \in \mathcal{R}_1\left(\mu_1^{E1}, \mu_1^{E2}\right) \times \mathcal{R}_2\left(\mu_2^{E1}, \mu_2^{E2}\right)$ such that condition (ii) in the Strong Revelation Principle (Theorem 2) is satisfied.

Since (ρ_1, ρ_2) are in $\mathcal{R}_1\left(\mu_1^{E1}, \mu_1^{E2}\right) \times \mathcal{R}_2\left(\mu_2^{E1}, \mu_2^{E2}\right)$, $\rho_j^n(j) = \mu_j^{En}$ for $n = 1, 2$. However, $\rho_j^n(j)$ may not be equal to μ_j^{En} for some $n \geq 3$. Because of that, condition (i) in the Strong Revelation Principle (Theorem 2) may not be satisfied.

We construct $\bar{\rho}_1$ and $\bar{\rho}_2$ that satisfy both conditions (i) and (ii) in the Strong Revelation Principle (Theorem 2) as follows. For each j, let $\bar{\rho}_j^1 = \rho_j^1$ and $\bar{\rho}_j^2 = \rho_j^2$. Because ρ_j^1 and ρ_j^2 are EPIC, $\bar{\rho}_j^1$ and $\bar{\rho}_j^2$ are EPIC. For all $n \geq 3$, all $(\omega_1, \ldots, \omega_n) \in \Omega^n$, $\bar{\rho}_j^n$ satisfies

$$\bar{\rho}_j^n\left[(j, \omega_h)_{h=1}^n\right] = \bar{\rho}_j^n\left[(k, \omega_i), (j, \omega_h)_{h \neq i}\right] = \rho_j^n\left[(j, \omega_i)_{h=1}^n\right], \ \forall i, \tag{63}$$

$$\bar{\rho}_j^n\left[(k, \omega_h)_{h=1}^n\right] = \bar{\rho}_j^n\left[(j, \omega_i), (k, \omega_h)_{h \neq i}\right] = \rho_j^n\left[(k, \omega_i)_{h=1}^n\right], \ \forall i. \tag{64}$$

(63) and (64) show that $\bar{\rho}_j^n$ ignores the report in $\mathcal{J}$ by an agent when it is not the same as the reports from the remaining agents. Since ρ_j^n is EPIC, $\bar{\rho}_j^n$ for $n \geq 3$ are EPIC and hence $\bar{\rho}_j = \left\{\bar{\rho}_j^n\right\}_{n=1}^I$ is EPIC for all j. Furthermore, $\bar{\rho}_k(j) = \rho_k(j)$ for all k, j. Therefore, $\bar{\rho}_1$ and $\bar{\rho}_2$ that satisfy both conditions (i) and (ii) in Theorem 2.