Complete Proofs of Theorems 1 and 2

in Menu Theorems for Bilateral Contracting

We start with some basic definitions. For any subset $Z \subset \mathcal{A}_i^j$, define the mapping $\tau(Z) : \mathcal{A}_i^j \to \mathcal{A}_i^j$ such that

$$\tau(Z)(\alpha_i^j) = \begin{cases} \alpha_i^j & \alpha_i^j \in Z \\ \widetilde{\alpha}_i^j & \alpha_i^j \notin Z \end{cases}$$

where $\tau(Z)(\alpha_i^j)$ is the incentive contract assigned in $\tau(Z)$ when agent i chooses α_i^j and $\widetilde{\alpha}_i^j$ is an arbitrary element of Z. Let $\gamma_i^j(\mathcal{C})$ be the image of γ_i^j. Consider the map, $\psi_i^j : \Gamma_i^j \to \overline{\Gamma}_i^j$ satisfying

$$\psi_i^j : \gamma_i^j \mapsto \overline{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$$

This map converts a mechanism γ_i^j into the menu of alternatives that γ_i^j provides. It is possible that two or more mechanisms provide the same menu of alternatives, so ψ_i^j is a many-to-one mapping with the inverse correspondence ξ_i^j. ψ^j is the mapping satisfying

$$\psi^j : (\gamma_1^j, \cdots \gamma_I^j) \mapsto (\overline{\gamma}_1^j(\cdot), \cdots, \overline{\gamma}_I^j(\cdot))$$

where $\overline{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$ for all $i \in \mathcal{I}$. Let ξ^j be the inverse correspondence of ψ^j. Finally, ψ is the mapping satisfying

$$\psi : (\gamma_1^1, \cdots \gamma_I^J) \mapsto (\overline{\gamma}_1^1(\cdot), \cdots, \overline{\gamma}_I^J(\cdot))$$

where $\overline{\gamma}_i^j(\cdot) = \tau(\gamma_i^j(\mathcal{C}))(\cdot)$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$. Let ξ be the inverse correspondence of ψ.

Given a collection of mechanisms $\gamma \in \Gamma$ and agent i's valuation $\omega_i \in \Omega_i$, the payoff for agent i associated with incentive contracts $\alpha_i = (\alpha_i^1, \cdots, \alpha_i^J) \in \mathcal{A}_i$ and $e_i \in E_i$ is given by

$$y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot,\cdot|\gamma,\cdot)) =$$

$$\int u(a(e), e, \omega_i) d\alpha_i(a_i) d\alpha_{-i}(a_{-i}) d\pi_{-i}(\alpha_{-i}, e_{-i}|\gamma, \omega_{-i}) dF(\omega_{-i}|\omega_i)$$

Since agent i's payoffs depend on a_{-i} and e_{-i}, y does depends on $\pi_{-i}(\cdot,\cdot|\gamma,\cdot)$. The equilibrium payoff for agent i is then

$$U(\widetilde{m}, \gamma, \omega_i) = \tag{1}$$

$$\int y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot,\cdot|\gamma,\cdot)) d\pi_i(\alpha_i, e_i|\gamma, \omega_i) =$$

$$\underset{\alpha_i, e_i}{Max}\{y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot,\cdot|\gamma,\cdot)) : \alpha_i^j \in \gamma_i^j(\mathcal{C}_i^j)\ \forall j \in \mathcal{J},\ e_i \in E_i\}$$

The second equality holds because any array of incentive contracts and any effort that agent i chooses with positive probability must maximize her payoff.

0.1 Proof of Theorem 1

Proof. Fix a pure strategy equilibrium $\widetilde{\gamma} = (\widetilde{\gamma}_1^1, \cdots, \widetilde{\gamma}_I^J) \in \Gamma$ given a continuation equilibrium $\widetilde{m}$ relative to Γ. Given the continuation equilibrium $\widetilde{m}$, each collection of mechanisms $\gamma = (\gamma_1^1, \cdots, \gamma_I^J)$ in Γ is transformed into a collection of menus with the mapping ψ. For all $\overline{\gamma} = (\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J) \in \overline{\Gamma}$, define $(G(\overline{\gamma}_1^1), \cdots, G(\overline{\gamma}_I^J))$ such that for all $j \in \mathcal{J}$ and all $i \in \mathcal{I}$

$$G(\overline{\gamma}_i^j) = \begin{cases} \widetilde{\gamma}_i^j & \text{if } \widetilde{\gamma}_i^j \in \xi_i^j(\overline{\gamma}_i^j) \\ \overline{\xi}_i^j(\overline{\gamma}_i^j) & \text{otherwise} \end{cases}$$

where $\overline{\xi}_i^j(\overline{\gamma}_i^j)$ is an arbitrary mechanism in $\xi_i^j(\overline{\gamma}_i^j)$. Now we can specify the continuation equilibrium relative to $\overline{\Gamma}$.

The continuation strategy for agent i is constructed as follows: for all $\overline{\gamma} = (\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J) \in$

$\overline{\Gamma}$ and all $\omega_i \in \Omega_i$

$$\widehat{q}_i(\cdot,\cdot|\overline{\gamma},\omega_i) = \pi_i(\cdot,\cdot|G(\overline{\gamma}_1^1),\cdots,G(\overline{\gamma}_I^J),\omega_i) \tag{2}$$

Principal j's strategy is $(\widehat{\gamma}_1^j,\cdots,\widehat{\gamma}_I^j) = (\psi_1^j(\widetilde{\gamma}_1^j),\cdots,\psi_I^j(\widetilde{\gamma}_I^j))$.

First, we need to prove that continuation strategies described above constitute a continuation equilibrium relative to $\overline{\Gamma}$. Consider the payoff for agent i after she chooses α_i and e_i when $(\overline{\gamma}_1^1,\cdots,\overline{\gamma}_I^J)$ is the collection of menus that principals offer and agent i's valuation is ω_i. (2) implies that this payoff is

$$y_i(\alpha_i,e_i,\omega_i,\pi_{-i}(\cdot,\cdot|\gamma,\cdot)) =$$

$$\int u(a(e),e,\omega_i)d\alpha_i(a_i)d\alpha_{-i}(a_{-i})d\pi_{-i}(\alpha_{-i},e_{-i}|G(\overline{\gamma}_1^1),\cdots,G(\overline{\gamma}_I^J),\omega_{-i})dF(\omega_{-i}|\omega_i)$$

From the definition of the continuation equilibrium $\widetilde{m}$, any array of incentive contracts and any effort in the support of $\pi_i(\cdot|G(\overline{\gamma}_1^1),\cdots,G(\overline{\gamma}_I^J),\omega_i)$ must maximize agent i's payoff conditional on $\{G(\overline{\gamma}_1^1),\cdots,G(\overline{\gamma}_I^J),\omega_i\}$. In continuation equilibrium in the original game, the payoff for agent i is equal to

$$\underset{\alpha_i,e_i}{Max}\{y_i(\alpha_i,e_i,\omega_i,\pi_{-i}(\cdot,\cdot|G(\overline{\gamma}_1^1),\cdots,G(\overline{\gamma}_I^J),\cdot)) : \alpha_i^j \in G(\overline{\gamma}_i^j)(\mathcal{C})\ \forall j \in \mathcal{J},\ e_i \in E_i\}$$

For each $i \in \mathcal{I}$ and each $j \in \mathcal{J}$, the choice set of incentive contracts $\overline{\gamma}_i^j(\mathcal{A}_i^j)$ provided by the menu $\overline{\gamma}_i^j$ is equal to $G(\overline{\gamma}_i^j)(\mathcal{C})$ provided by the mechanism $G(\overline{\gamma}_i^j)$. It is then immediate that the continuation strategy $\widehat{q}_i$ is optimal for each $i \in \mathcal{I}$. Therefore, the array of continuation strategies $\widehat{q} = (\widehat{q}_1,\cdots,\widehat{q}_I)$ constitutes a continuation equilibrium relative to $\overline{\Gamma}$. Moreover, it shows

$$U(\widehat{q},\widehat{\varepsilon},\overline{\gamma}_1^1,\cdots,\overline{\gamma}_I^J,\omega_i) =$$

$$\underset{\alpha_i, e_i}{Max}\{y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot,\cdot|G(\overline{\gamma}_1^1), \cdots, G(\overline{\gamma}_I^J), \cdot)) : \alpha_i^j \in G(\overline{\gamma}_i^j)(\mathcal{C}_i^j)\ \forall j \in \mathcal{J},\ e_i \in E_i\} =$$

$$\int y_i(\alpha_i, e_i, \omega_i, \pi_{-i}(\cdot,\cdot|\gamma,\cdot))d\pi_i(\alpha_i, e_i|G(\overline{\gamma}_1^1), \cdots, G(\overline{\gamma}_I^J), \omega_i) =$$

$$U(\widetilde{m}, \widetilde{\varepsilon}, G(\overline{\gamma}_1^1), \cdots, G(\overline{\gamma}_I^J), \omega_i)$$

If $(\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J)$ is equal to $(\widehat{\gamma}_1^1, \cdots, \widehat{\gamma}_I^J)$,

$$U(\widehat{q}, \widehat{\varepsilon}, \widehat{\gamma}_1^1, \cdots, \widehat{\gamma}_I^J, \omega_i) =$$

$$U(\widetilde{m}, \widetilde{\varepsilon}, G(\widehat{\gamma}_1^1), \cdots, G(\widehat{\gamma}_I^J), \omega_i) =$$

$$U(\widetilde{m}, \widetilde{\varepsilon}, \widetilde{\gamma}_1^1, \cdots, \widetilde{\gamma}_I^J, \omega_i)$$

Therefore, the equilibrium payoffs for agent i in the original game is reproduced when principals offer menus $(\widehat{\gamma}_1^1, \cdots, \widehat{\gamma}_I^J)$.

Suppose that principal j offers $\widehat{\gamma}^j$ given $\widehat{\gamma}^{-j}$. The payoff for principal j is

$$V^j(\widehat{\gamma}^j, \widehat{\gamma}^{-j}, \widehat{q}) =$$

$$\int v^j(a(e), e, \omega)d\alpha(a)d\pi(\alpha, e|G(\widehat{\gamma}_1^1), \cdots, G(\widehat{\gamma}_I^J), \omega)dF(\omega) =$$

$$\int v^j(a(e), e, \omega)d\alpha(a)d\pi(\alpha, e|\widetilde{\gamma}_1^1, \cdots, \widetilde{\gamma}_I^J, \omega)dF(\omega) =$$

$$V^j(\widetilde{\gamma}^j, \widetilde{\gamma}^{-j}, \widetilde{m})$$

Therefore, the equilibrium payoff for principal j is preserved when principals offer menus $(\widehat{\gamma}_1^1, \cdots, \widehat{\gamma}_I^J)$. Suppose that principal j deviates to some other array of menus, say $\gamma^j \in \overline{\Gamma}^j$.

$$V^j(\gamma^j, \widehat{\gamma}^{-j}, \widehat{q}) =$$

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | G(\gamma^j), G(\widehat{\gamma}^{-j}), \omega) dF(\omega) =$$

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e | \overline{\xi}^j(\gamma^j), \widetilde{\gamma}^{-j}, \omega) dF(\omega) =$$

$$V^j(\overline{\xi}^j(\gamma^j), \widetilde{\gamma}^{-j}, \widetilde{m}, \widetilde{\varepsilon}) \leq$$

$$V^j(\widetilde{\gamma}^j, \widetilde{\gamma}^{-j}, \widetilde{m}, \widetilde{\varepsilon})$$

Therefore, $(\widehat{\gamma}_1^1, \cdots, \widehat{\gamma}_I^J) \in \overline{\Gamma}$ is the pure strategy equilibrium given the continuation equilibrium $\widehat{q}$ relative to $\overline{\Gamma}$ that preserves the equilibrium payoffs associated with a pure strategy equilibrium $(\widetilde{\gamma}_1^1, \cdots, \widetilde{\gamma}_I^J) \in \Gamma$ given a continuation equilibrium $\widetilde{m}$ relative to Γ.

0.2 Proof of Theorem 2

We start with some basic definitions. Let us take two models for bilateral contracting Γ and $\overline{\Gamma}$ such that $\Gamma \succcurlyeq \overline{\Gamma}$. Let the associated continuation equilibria be $\widetilde{m}$ and $\widehat{q}$ respectively. $\pi(\cdot, \cdot | \gamma, \omega)_{\widetilde{m}}$ and $\pi(\cdot, \cdot | \overline{\gamma}, \omega)_{\widehat{q}}$ denotes the joint probability distributions on $\mathcal{A} \times E$ induced by the two continuation equilibria when the collections of mechanisms are $\gamma = (\gamma_1^1, \cdots \gamma_I^J) \in \Gamma$ and $\overline{\gamma} = (\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J) \in \overline{\Gamma}$ respectively.[1] $\widetilde{m}$ is said to *extend* $\widehat{q}$ if there is an embedding $\eta_i^j : \overline{\Gamma}_i^j \to \Gamma_i^j$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$ such that for all $\overline{\gamma} = (\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J) \in \overline{\Gamma}$

$$\pi(\cdot, \cdot | \overline{\gamma}, \omega)_{\widehat{q}} = \pi(\cdot, \cdot | \eta(\overline{\gamma}), \omega)_{\widetilde{m}}$$

where $\eta(\overline{\gamma}) = (\eta_1^1(\overline{\gamma}_1^1), \cdots, \eta_I^J(\overline{\gamma}_I^J))$, $\pi(\cdot, \cdot | \overline{\gamma}, \omega)_{\widehat{q}} = \pi_1(\cdot, \cdot | \overline{\gamma}, \omega_I)_{\widehat{q}_1} \times \cdots \times \pi_1(\cdot, \cdot | \overline{\gamma}, \omega_I)_{\widehat{q}_1}$ and $\pi(\cdot, \cdot | \eta(\overline{\gamma}), \omega)_{\widetilde{m}} = \pi_1(\cdot, \cdot | \eta(\overline{\gamma}), \omega_1)_{\widetilde{m}_1} \times \cdots \times \pi_I(\cdot, \cdot | \eta(\overline{\gamma}), \omega_I)_{\widetilde{m}_I}$. It generalizes the idea behind direct mechanisms in the single principal problem such that principals explore more complex mechanisms in Γ that are not provided by the model of competition

[1] I make the notation explicitly contingent on those strategies to highlight the difference of two models of competition.

specified in bilateral contracting game relative to $\overline{\Gamma}$.

Proof. The method of the proof is to transform deviations that lie outside of the range of η into menus that they provide and then change the continuation equilibrium associated with those menus to coincide with the original equilibrium. The mapping ψ_i^j is used to associate each mechanism $\gamma_i^j \in \Gamma_i^j$ with the corresponding menu $\overline{\gamma}_i^j = \tau(\gamma_i^j(\mathcal{C}))$ for all $i \in \mathcal{I}$ and all $j \in \mathcal{J}$.

$\widehat{q}_i(\cdot|\overline{\gamma}, \omega_i)$ maximizes agent i's payoff given the other players' strategies $\widehat{q}_{-i}$ when the collection of menus is $\overline{\gamma} = (\overline{\gamma}_1^1, \cdots, \overline{\gamma}_I^J)$ and agent i's valuation is ω_i. It follows that when the collection of mechanisms is $\gamma \in \Gamma$ and agent i's valuation is ω_i, a continuation strategy that can induce probability distribution $\pi_i(\cdot,\cdot|\psi(\gamma), \omega_i)_{\widehat{q}_i}$, maximizes agent i's payoff. Choose a continuation equilibrium $\widetilde{m}$ relative to Γ satisfying, for all $i \in \mathcal{I}$, all $\gamma \in \Gamma$, and all $\omega_i \in \Omega_i$

$$\pi_i(\cdot,\cdot|\gamma, \omega_i)_{\widetilde{m}_i} = \pi_i(\cdot,\cdot|\psi(\gamma), \omega_i)_{\widehat{q}_i}$$

Let η^j be the mapping satisfying that $\eta^j(\overline{\gamma}_1^j, \cdots, \overline{\gamma}_I^j) = (\eta_1^j(\overline{\gamma}_1^j), \cdots, \eta_I^j(\overline{\gamma}_I^j))$ for all $(\overline{\gamma}_1^j, \cdots, \overline{\gamma}_I^j) \in \overline{\Gamma}^j$. The strategy $\widetilde{\sigma}^j$ chosen by principal j is induced by the mapping η^j given $\widehat{\upsilon}^j$.

We begin with an equilibrium $(\widehat{\upsilon}, \widehat{q})$ relative to $\overline{\Gamma}$. The payoff for principal j who unilaterally deviates to some array of mechanisms $\gamma^j = (\gamma_1^j, \cdots, \gamma_I^j)$ outside of $\eta^j(\overline{\Gamma}^j)$ is given by

$$\int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e|\gamma^j, \eta^{-j}(\overline{\gamma}^{-j}), \omega)_{\widetilde{m}} \widetilde{\sigma}^{-j}(\eta^{-j}(\overline{\gamma}^{-j})) dF(\omega)$$

$$= \int v^j(a(e), e, \omega) d\alpha(a) d\pi(\alpha, e|\psi^j(\gamma^j), \overline{\gamma}^{-j}, \omega)_{\widehat{q}} \widehat{\upsilon}^{-j}(\overline{\gamma}^{-j}) dF(\omega)$$

$$\leq V^j(\widehat{\upsilon}^j, \widehat{\upsilon}^{-j}, \widehat{\gamma}^{-j}, \widehat{q})$$

$$= V^j(\widetilde{\sigma}^j, \widetilde{\sigma}^{-j}, \widetilde{m})$$

which proves that an equilibrium $(\widehat{v}, \widehat{q})$ relative to $\overline{\Gamma}$ is weakly robust.