General Competing Mechanism Games with Strategy-Proof Punishment

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April 22, 2022

Abstract

This paper studies a general competing mechanism game of incomplete information where each seller can offer a contract that determines a menu of any complex mechanisms conditional on buyers' messages, and then chooses a mechanism that he wants from the menu. The focus is the class of robust equilibria in which sellers punish a deviating seller with dominant strategy incentive compatible direct mechanisms. We show how to characterize the class of such equilibria. In applications, the number of sellers is endogenized given a number of buyers and fixed entry costs. In a large market, a seller's equilibrium ex-ante expected net profit is always equal to zero but any price in an interval can be supported as an equilibrium price. The equilibrium ratio of the number of buyers to the number of sellers is lowest at the monopoly price, and increases as the price moves farther away from it in either direction. (JEL C72, D47, D82)

*For their comments and suggestions, I thank Jiangtao Li (associate editor), two anonymous referees, and seminar attandants at various conferences. I benefitted from discussion with Andrea Attar and Siyang Xiong. I also thank Andrew Leal and Zvez Todorov for their excellent research assistance. I thankfully acknowledge financial support from the Social Sciences and Humanities Research Council of Canada. Address: Department of Economics, McMaster University, 1280 Main Street West, Hamilton, ON, CANADA. Email: hansj@mcmaster.ca

1 Introduction

Buyers are informed about terms of trade or prices offered by sellers in the market as they search for a better deal. Sellers often gather market information from buyers to determine his terms of trade in addition to eliciting their willingness to pay (i.e., payoff type). This is a prominent feature of a seller's web design in on-line markets. On-line sellers can keep track of buyers' search history based on html cookies, most of which are based on simple binary messages: Whether or not a buyer revisits a seller's web site, whether or not a buyer clicks a certain part of a seller's web site, etc. This type of information can reveal what buyers know about competing sellers' terms of trade (Board and Lu (2018), Peters (2015)). Despite the prevalent use of buyers' search behavior or their market information in practice, it is not easy to develop a tractable competition theory that reflects it, due to the potential complexity of market information (Epstein and Peters (1999), Yamashita (2010)).

This paper introduces a *general competing mechanism game* of incomplete information. In this game, buyers' payoff types are their private information and each buyer has to choose only one seller for trading as in the literature on competing auctions (e.g., McAfee (1993), Peters and Severinov (1997), Virag (2010), etc.). Sellers choose only reserve prices for their auctions in competing auctions and participating buyers are restricted to submit their bids. However, our game is more general in terms of applications, and importantly it can incorporate buyers' on-line search behavior, allowing buyers' communication to take place over two rounds.

Given a profile of contracts offered by sellers, each buyer privately sends a first-round message to each seller.[1] Each seller's contract specifies a menu of mechanisms conditional on all buyers' first-round messages and then the seller chooses a mechanism from the menu. The messages in the first round are similar to information that buyers leave (through html cookies) on sellers' web sites they visit. In this round, buyers do not need to select a seller. Sellers' web sites are then responsive in the sense that they can change a selling mechanism conditional on what are left in html cookies (e.g., the reserve price in an auction can be responsive to information in html cookies).

In the second round, after observing sellers' mechanisms, each buyer privately

[1] Attar, et al. (2018) and Attar, et al. (2021) study competing mechanism games where a buyer can communicate only with a seller she chooses for trading. However, the standard characterization of PBE in competing mechanism games do not rely on that restriction on communication. See the discussion in Section 2.2 of Epstein and Peters (1999).

sends a message only to a seller whom she selects (e.g., buyers submit their bids only to a seller's auction site they select after search). A seller's mechanism then specifies a profile of probabilities of action alternatives and monetary transfers to participating buyers conditional on their second-round messages. The nature of a seller's action can be general, and different depending on applications such as selling a private good with or without a capacity constraint, selling an (excludable) public good, etc.

Buyers adopt symmetric selection behavior in the sense that a buyer chooses one of the sellers with the same contracts with equal probability.[2] Therefore, sellers with the same contracts may end up with a different number of buyers, which can be thought of as endogenous frictions in the market. This paper does not attempt to characterize the set of all possible equilibrium allocations. Instead, it characterizes the set of equilibrium allocations that can be sustained when non-deviating sellers always choose dominant-strategy incentive compatible direct mechanisms when buyers' first-round messages reveal a deviation by a competing seller.

Dominant-strategy incentive compatibility of a direct mechanism is appealing particularly in a model where a buyer selects one seller for trading. It is very hard to work with Bayesian incentive compatible (BIC) direct mechanisms because Bayesian incentive compatibility is based on buyers' interim payoffs, which depends on buyers' selection strategies in a continuation equilibrium given a profile of mechanisms chosen by sellers. This implies that Bayesian incentive compatibility of a seller's direct mechanism depends on the mechanisms that all sellers choose. On the other hand, the dominant strategy incentive compatibility of a direct mechanism can be defined without reference to buyers' selection strategies or other sellers' mechanisms. For that reason, we focus on the set of equilibria where non-deviating sellers punish a deviating seller with dominant strategy incentive compatible (DIC) direct mechanisms. We call it a *strategy-proof punishment*.

We can show that in such equilibria, the lower bound of each seller j's equilibrium payoff is his minmax value where the min is taken over the other sellers' DIC direct mechanisms and the max is taken over j's BIC direct mechanisms conditional on the others' DIC direct mechanisms. However, one may wonder if and when we can restrict a deviating seller's deviation to DIC direct mechanisms without loss of generality. If buyers' types are one-dimensional, private and independent and payoff

[2]For simplicity of the notation, we assume that buyers are ex-ante identical and sellers are also identical. We can extend the results with (ex-ante) heterogeneous buyers and sellers but the notation is considerably heavier.

functions are linear, we can extend the BIC-DIC equivalence in Gershkov, et. al. (2013) to show that any BIC direct mechanism induced by a pair of (i) a deviating seller's mechanism and (ii) buyers' communication strategies can be replaced by a payoff-preserving DIC direct mechanism in a given continuation equilibrium. Using this BIC-DIC equivalence, we establish that every seller can restrict himself to offer a contract that specifies a menu of DIC direct mechanisms conditional on buyers' messages on and off the equilibrium path without loss of generality, even if a seller can offer a contract that assigns a set of any arbitrary mechanisms, not just DIC direct mechanisms. Therefore, the lower bound of each principal's equilibrium utility is his minmax value only over DIC direct mechanisms. We also show that the first-round messages in the non-deviators' equilibrium contracts can be as simple as binary (i.e., whether or not there is a deviating seller). As in Yamashita (2010), agents can truthfully reveal whether or not there is a deviating seller when there are three or more agents because a non-deviating seller can take the same report from a strict majority of agents as the true information about deviating sellers.

This paper considers two applications where sellers can each produce a unit of a homogenous private good at a constant marginal cost and each buyer has a unit demand with private valuation: one without a seller's capacity constraint and the other with. We first consider the case where sellers have no capacity constraint. Non-deviating sellers can lower their price down to their marginal cost upon a competing seller's deviation reported by buyers in the first around of communication. This implies that a deviating seller must incur loss (i.e., lower his price below the marginal cost) in order to attract buyers upon deviation to any arbitrary mechanism. Therefore, the lower bound of a seller's equilibrium profit can be as low as a seller's reservation profit. If the hazard rate of the buyer's valuation is non-decreasing, the upper bound of a seller's equilibrium profit is reached at the monopoly price, i.e., the buyer's valuation that makes her virtual valuation equal to zero. This analysis is based on a fixed number of buyers and sellers.

However, sellers may freely enter the market by incurring a fixed entry cost C. A seller has a belief on the trading price x in the market. Given the number of buyers N, sellers will keep entering the market as long as they can earn non-negative ex-ante expected profit net of the fixed entry cost. Given x and N, the equilibrium number of sellers J is determined at the point where one additional seller in the market yields a negative ex-ante expected profit net of the fixed entry cost. Let a buyer's valuation follow a probability distribution F. Each seller's equilibrium ex-ante expected net profit is an equal split of the total profit that a single seller could earn if he had sold the product to buyers at price x alone - $\frac{Nx(1-F(x))}{J} - C$. Further, there is a range of equilibrium prices.

In a finite market, each seller's equilibrium ex-ante expected net profit $\frac{Nx(1-F(x))}{J} - C$ can be positive. However, as $N \to \infty$, the equilibrium ratio of the number of buyers to the number of sellers makes each seller's equilibrium ex-ante expected net profit converges to zero. Despite this, the multiplicity of equilibrium prices does not go away even in a large market. One interesting equilibrium feature is that in both finite and large markets, the largest number of sellers in the market occurs when the trading price is equal to the monopoly price, and the number of sellers in the market decreases as the trading price moves farther away from the monopoly price in either direction.

Finally, we show that this large market result in the case with no capacity constraint holds even when each seller faces a capacity constraint. As in a canonical environment for competing auctions (Burguet and Sákovics (1999), Han (2015), McAfee (1993), Peters and Severinov (1997), Peters (1997), Virag (2010)), we consider a large market where the number of buyers is taken to infinity with a fixed ratio of buyers to sellers. There is a range of equilibrium reserve prices. Given an equilibrium reserve price, the equilibrium ratio of the number of buyers to the number of sellers is determined at the point where a seller's ex-ante expected net profit is equal to zero. The equilibrium ratio of the number of buyers to sellers is lowest when the reserve price is equal to the monopoly reserve price and increases as the reserve price moves farther away from the monopoly price.

1.1 Related literature

In standard competing mechanism games (e.g., Epstein and Peters (1999), Yamashita (2010)), a seller (principal) is only allowed to offer contracts that delegate his allocation decision to buyers (agents). In other words, buyers' messages completely determine the allocation decision. Szentes (2009) shows why it is restrictive for a seller to focus only on such contracts in a game of complete information. If a principal is restricted to delegate his action choice, non-deviating principals can punish the deviating seller by choosing their actions conditional on the deviator's action. This makes the lower bound of a principal's equilibrium payoff equal to the maxmin value of his payoff over actions.

On the other hand, if a principal can offer a contract in which agents' messages assign a set of actions from which the principal can choose, he can deviate to a contract that assigns the set of all possible actions regardless of buyers' messages. Upon the principal's deviation to such a contract, he can choose an action that maximizes his payoff among all possible actions, conditional on non-deviators' action choice. This raises the lower bound of a principal's equilibrium payoff to the minmax value of his payoff over actions.

Our paper extends Szentes' approach in a game of incomplete information by allowing a principal to offer a contract in which agents' first-round messages determines a set of mechanisms from which the principal chooses. Focusing on the set of equilibrium allocations with strategy-proof punishment, we show that the lower bound of a principal's equilibrium utility is the minmax value over direct mechanisms instead of the maxmin value.

Our model also differs from the standard competing mechanism game in two other fronts. First, it allows principals to ask agents to send messages over two rounds. Second, buyers' first-round messages in a contract only determine a set of direct mechanisms from which a principal can eventually choose, as long as the set is not a singleton.

Attar, et al. (2021) consider a single round of communication from an agent to only a principal she select.[3] They provide an example of complete information, where, if there is a principal (e.g., seller), say j, who is restricted to offering only direct mechanisms, while others can offer more complex mechanisms, principal j's equilibrium payoff can be even lower than the lower bound of his equilibrium payoff in the competing mechanism game with only direct mechanisms. This can be seen as a failure of the Revelation Principle in a model where an agent can select only one principal and communicates with him only. Their logic goes through for the competing mechanism games in Szentes (2009) as well. Notably, our paper establishes the lower bound of a principal's equilibrium payoff, in terms of direct mechanisms, that can be supported with strategy-proof punishment with *no restrictions* on a principal's ability to offer any complex contracts and in an environment where market information can be

[3]In Attar, et al. (2010), Epstein and Peters (1995) and our paper, agents have to select one principal for trading. In Epstein and Peters (1995) and our paper, agents can communicate with principals they do not choose, whereas in Attar, et al. (2021), agents can communicate only with a principal they select. In Szentes (2009) and Yamashita (2010), agents do not need to select a principal and they communicate with all principals.

freely transmitted to principals through the non-exclusive first-round communication with three or more agents.[4] This is the focal point of our analysis in Sections 4 and 5.

In our model, a principal has full commitment power. That is, if he wants, a principal can offer a contract that fully specifies his direct mechanism conditional on agents' messages. In other words, the domain of a contract in our paper is the set of all non-empty subsets of the set of all mechanisms. Therefore, if a principal chooses a contract where agents' messages leave a set of multiple mechanisms with the principal, it is not because he has no commitment power but because it is strategically better for him to do so. This is the key difference from the approach in the literature on the mechanism design by a single principal with limited commitment (e.g., Bester and Strausz (2000, 2001, 2007)), which assumes that a principal simply cannot restrict the set of allocation decisions he can choose, conditional on agents' messages.

In fact, the Revelation Principle generally does not hold for mechanism design by a single principal with limited commitment (Bester and Strausz (2000)). However, it does hold in our model when there is only one principal. With only one principal, the set of the first-round messages can be simply a singleton (i.e., not necessary) and the principal can focus on the incentive compatible direct mechanisms even if he can offer any general contract allowed in our model. What we show in this paper is that allowing for general contracts with two- round communication makes a difference in the model of incomplete information with multiple principals in that it raises the lower bound of a principal's equilibrium payoff.

Our paper complements Han (2021) who proposes a general competing mechanism game with a single round of communication in the environment where agents (e.g., buyers) can trade with all principals (e.g., sellers). Given a notion of robust equilibrium proposed in Han (2021), the separability of agents' payoff functions with respect

[4]In our model, the market information can be freely transmitted to principals through the non-exclusive first-round communication. Therefore, the second-round communication from an agent only to a principal whom she selects can be thought of as reporting her remaining private information, i.e., her payoff type.

In another example, Attar, et al. (2021) consider the environment where each agent has a most preferred principal regardless of principals' actions and the other agents' participation decisions. The example shows that in this environment, a principal's payoff above the lower bound identified with no restrictions on mechanisms principals can offer cannot be an equilibrium payoff. However, our model considers a different environment in that (i) there are no such dominant participation decisions for agents because principals are equally preferable by an agent if their actions are the same and (ii) agents can freely communicate with any principal even before they select a principal for trade.

to principals' actions leads to the full characterization of equilibrium allocations in terms of incentive compatible direct mechanisms. They are also free from Szentes' Critique. Our paper proposes a general competing mechanism game with two rounds of communication in an environment where an agent eventually has to choose a single seller for trading, and provides (i) the equilibrium characterization supportable with strategy-proof punishments and (ii) comparative statics in applications.

2 Preliminaries

If necessary, we assume that a set X represents a compact metric space. Where only a weaker structure is needed, it will be made explicit. When a measurable structure is necessary, the corresponding Borel σ - algebra is used.

J principals ($J \geq 2$, e.g., sellers) compete in a market with N agents ($N \geq 3$,e.g., buyers). Throughout the paper, we use the terms *sellers* and *buyers* instead of principals and agents for ease of exposition, but the model can capture other principal/agent applications. Buyers are ex-ante identical in that each buyer's type is independently drawn from a probability distribution F with support $X = [\underline{x}, \bar{x}] \subset \mathbb{R}_+$. Each buyer's type is not observed by anyone else.[5] When a seller of type w makes an allocation decision α for buyers who select him, $v(\alpha, w)$ is the seller's payoff and $u(\alpha, x)$ is the payoff for a buyer of type x who selects him. We assume that a seller's type is publicly observable.

For most parts of the paper, we impose structure on a seller's allocation decision α and players' payoff functions v and u that are needed for equivalence between Bayseian incentive compatibility and dominant strategy incentive compatibility. However, with general payoff functional forms, $v(\alpha, w)$ and $u(\alpha, x)$, we can characterize the set of robust equilibrium allocations, in terms of incentive compatible direct mechanisms, when sellers punish a deviating seller with DIC direct mechanism (See Section 5.1).

Each seller's allocation decision is to choose (i) one of his action alternatives from the finite set, $\mathcal{K} = \{1, \ldots, K\}$, and (ii) monetary transfers to buyers who select him. A buyer's payoff depends on the action taken by the seller she selects and monetary transfer. The payoff for a buyer of type x associated with choosing a seller who takes action alternative k is

$$b^k x + g^k + t, \tag{1}$$

[5]We use masculine pronouns for sellers and feminine pronouns for buyers.

where $b^k \geq 0$ and $g^k \in \mathbb{R}$ for all $k \in \mathcal{K}$ and $t \leq 0$ is a monetary transfer to the buyer.[6] For example, k is the identity of the winning bidder in an auction environment. In this case, bidder n has the following preference parameter values: $b^k = 1$ if $k = n$, $b^k = 0$ otherwise; $g^k = 0$ for all k. If a buyer does not choose any seller to trade with, she receives her reservation payoff, which is equal to zero.

A seller's payoff depends on his choice of action alternative k and monetary transfer ℓ;

$$a^k w + y^k + \ell, \tag{2}$$

where $a^k, y^k \in \mathbb{R}$ for all $k \in \mathcal{K}$, $w \in \mathbb{R}$. We assume that $\max_k[a^k w + y^k] \geq 0$. A seller's type w is publicly known and the same to every seller. The homogeneity of sellers and buyers is for simplicity but it can be relaxed (See Section 5.2).

For an auction environment, we can set $\mathcal{K} = \{1, \ldots, N, N+1\}$ so that, given N potential bidders, alternative k implies that the winning bidder is bidder k if $k \leq N$, but $k = N + 1$ means that the seller retains the object. The seller has then the following preference parameter values: $a^k = 0$ for $k \leq N$ and $a^k = 1$ for $k = N + 1$; $y^k = 0$ for all k. w represents the value of the object to the seller. A seller's reservation payoff is equal to zero. We impose the *budget balance* condition for each seller so that the sum of ℓ and monetary transfers t to buyers who choose him is equal to zero. This implies that $\ell \geq 0$. Non-positive monetary transfers to a buyer and non-negative monetary transfers to a seller reflect limited liability on both buyers and sellers.

Gershkov, et al (2013) established the BIC-DIC equivalence with the payoff functions in (1) and (2). While g^k and y^k are equal to zero in an auction environment, they can be non-zero when alternative k provides common utilities to the seller and all buyers who select him in addition to individual specific utilities. To allow for such possibility, we also follow the same payoff functions considered in Gershkov, et al (2013)

[6]We consider the cases where a buyer (agent) pays a positive amount of money to a seller (principal). Because t is a monetary transfer to a buyer (agent), it means that t is assumed to be non-positive. The assumption of $t \leq 0$ essentially implies that a seller (principal) cannot make a positive monetary transfer to a buyer. This is what we usually observe in practice. We can assume $t \geq 0$ for the case where buyers are principals and sellers are agents.

3 Feasible Allocations

An (interim) allocation should tell us how each seller's action and monetary transfers depend on the participating buyers' types and how each buyer selects a sellers. In that sense, an allocation is characterized by (a) an array of direct mechanisms offered by sellers and (b) the buyer's selection behavior that decides how to select a seller among J sellers given an array of direct mechanisms.

If a buyer does not select a seller, he treats her type as x°. Let $\bar{X} = X \cup \{x^\circ\}$. For any given seller, $\mathbf{x} \in \bar{X}^N$ can conveniently characterize the type profile of the buyers who select him. Let a seller's direct mechanism μ be denoted by $\{q^1, \ldots, q^K, t\}$, where $q^k : \bar{X}^N \to [0,1]$ determines the probability of alternative k as a function of buyers' type messages and $t : \bar{X}^N \to \mathbb{R}_-$ determines an amount of monetary transfer to a buyer as a function of buyers' type messages.

We assume that mechanisms are *anonymous* with respect to buyers so that they cannot distinguish among different buyers except on the basis of messages sent by buyers. Sellers are also not distinguished except on the basis of their mechanisms (and contracts). We are interested in a (*symmetric*) *allocation* where (i) buyers choose sellers with equal probability as long as their mechanisms are the same and (ii) every seller's direct mechanism is identical.

We first construct a buyer's selection behavior given an array of direct mechanisms when one seller offers μ' and all the other sellers offer μ. Let $\pi(\mu', \mu)(x) \in [0,1]$ denote the probability with which each buyer of type x selects the seller who offers μ' when all the other sellers offer μ. Define $z(\pi(\mu', \mu))(x)$ as follows

$$z(\pi(\mu', \mu))(x) := 1 - \int_x^{\bar{x}} \pi(\mu', \mu)(s) dF. \tag{3}$$

The term $z(\pi(\mu', \mu))(x)$ is the probability that a buyer either has her type below x as a participant of μ' or selects any other seller whose mechanism is μ.

An array of BIC (Bayesian incentive compatible) direct mechanisms (μ', μ) chosen by sellers defines a subgame that buyers play.[7] A (truthful symmetric) *continuation equilibrium* at a subgame (μ', μ) can be characterized by (a) every buyer's selection strategy $\pi(\mu', \mu)$ and (b) Bayesian incentive compatibility of direct mechanisms.[8] Let

[7] (μ', μ) means that one principal offers μ' and all the other principals offer μ.

[8] In a (symmetric) continuation equilibrium at a subgame (μ', μ), an agent of type x selects a principal whose mechanism is μ with probability $\frac{1-\pi(\mu',\mu)(x)}{J-1}$ if she selects a principal with μ.

$\Pi(\mu', \mu)$ be the set of all possible continuation equilibria (i.e., the set of all optimal selection strategies that buyers choose in all continuation equilibria at a subgame (μ', μ)). Given $\mu' = \{\acute{q}^1, \ldots, \acute{q}^K, \acute{t}\}$, we can derive the reduced-form direct mechanism $\{\acute{Q}^1, \ldots, \acute{Q}^K, \acute{T}\}$ such that, for all $x \in X$,

$$\acute{Q}^k(x) := \int_{\underline{x}}^{\bar{x}} \cdots \int_{\underline{x}}^{\bar{x}} \acute{q}^k(x, s_2, \ldots, s_I) dz(\pi(\mu', \mu))(s_2) \ldots dz(\pi(\mu', \mu))(s_I), \tag{4}$$

$$\acute{T}(x) := \int_{\underline{x}}^{\bar{x}} \cdots \int_{\underline{x}}^{\bar{x}} \acute{t}(x, s_2, \ldots, s_I) dz(\pi(\mu', \mu))(s_2) \ldots dz(\pi(\mu', \mu))(s_I). \tag{5}$$

The interim expected payoff for a buyer of type x associated with reporting their true type upon selecting the seller whose direct mechanism is μ' is

$$U_J(\mu', \mu, \pi, x) := \sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x) + \acute{T}(x). \tag{6}$$

μ' is Bayesian incentive compatible (BIC) given $\pi(\mu', \mu)$ if for all $x, x' \in X$

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x) + \acute{T}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) \acute{Q}^k(x') + \acute{T}(x') \tag{7}$$

As shown in (7) above, incentive compatibility is imposed only for all $x, x' \in X$ but not $\bar{X}$, i.e., truth telling is optimal only for those buyers participating in the seller's mechanism. If a buyer does not select the seller, we treat her effective type as x° in $\bar{X}$ and there is no type reporting for her. The ex-ante expected payoff for the seller whose direct mechanism is μ' is[9]

$$\Phi_J(\mu', \mu, \pi) := \sum_{k \in \mathcal{K}} \int_{\underline{x}}^{\bar{x}} (a^k w + y^k) \acute{Q}^k(s_1) dz(\pi(\mu', \mu))(s_1) - N \int_{\underline{x}}^{\bar{x}} \acute{T}(s_1) dz(\pi(\mu', \mu))(s_1). \tag{8}$$

We define a (symmetric) feasible allocation as follows.

Definition 1 *(μ, π) is a (symmetric) feasible allocation if (i) μ is BIC given $\pi(\mu, \mu)$, (ii) a buyer chooses each seller according to $\pi(\mu, \mu)(x) = \frac{1}{J}$ whenever $\pi(\mu, \mu)(x) > 0$. (iii) for all $x \in X$, $\pi(\mu, \mu)(x) > 0$ if $U_J(\mu, \mu, \pi, x) \geq 0$, and (iv) $\Phi_J(\mu, \mu, \pi) \geq 0$.*

[9]The subscript in $\Phi_J(\mu', \mu, \pi)$ and $U_J(\mu', \mu, \pi, x)$ denotes the number of principals in the market.

In a feasible allocation, conditions (i) and (ii) imply that buyers must play a continuation equilibrium in the subgame where all sellers' mechanisms are identical. Conditions (iii) and (iv) imply that a feasible allocation must be *individually rational* for both buyers and sellers. Let Z be the set of all feasible allocations.

We are first interested in the set of a seller's ex-ante expected payoffs associated with all possible feasible allocations in Z. It cannot be lower than a seller's reservation payoff, which is zero. Therefore, the minimum of a seller's ex-ante expected payoff is zero. Given symmetry in allocation, the maximum of a seller's ex-ante expected payoff can be derived by solving a joint payoff maximization problem.

There may be a buyer who decides not to choose any seller when the direct mechanism μ is chosen by all sellers. In that case, we assume that a buyer first selects one of the sellers with equal probability $1/J$ and sends the type message x°. Therefore, in the joint payoff maximization problem, we fix $\pi(\mu,\mu)(x) = \frac{1}{J}$ for all $x \in X$. Then,

$$z(\pi(\mu,\mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x \in X. \tag{9}$$

Given a direct mechanism $\mu = \{q^1, \ldots, q^K, t\}$, let $\{Q^1, \ldots, Q^K, T\}$ be the reduced-form direct mechanism that is derived according to (4) and (5) with (9). Then, the joint payoff maximization problem is

$$\max_{\mu} \Phi_J(\mu, \mu, \pi) \tag{10}$$

subject to

$$\begin{aligned}
\text{(IC)} \quad \sum_{k\in\mathcal{K}}(b^k x + g^k)Q^k(x) + T(x) &\geq \sum_{k\in\mathcal{K}}(b^k x + g^k)Q^k(x') + T(x') \ \forall x, x' \in X,\\
\text{(IR)} \quad \sum_{k\in\mathcal{K}}(b^k x + g^k)Q^k(x) + T(x) &\geq 0 \ \forall x \in X,
\end{aligned}$$

and $q^k(\mathbf{x}) \geq 0$ and $\sum_{k=1}^{K} q^k(\mathbf{x}) \leq 1$ for all $\mathbf{x} \in \bar{X}^N$. We assume that a solution, denoted by $\bar{\mu} = \{\bar{q}^1, \ldots, \bar{q}^K, \bar{t}\}$, to the joint payoff maximization problem exists and let $\bar{\phi}_J := \Phi_J(\bar{\mu}, \bar{\mu}, \pi)$. Let Φ_J^* be the set of the seller's feasible ex-ante expected payoffs:

$$\Phi_J^* := \{\phi \in \mathbb{R} : \phi = \Phi_J(\mu, \mu, \pi) \ \forall\, (\mu, \pi) \in Z\}.$$

Lemma 1 $\Phi_J^* = [0, \bar{\phi}_J]$.

Proof. It is clear that a seller's ex-ante expected payoff associated with a feasible allocation cannot be less than zero and greater than $\bar{\phi}_J$. We only need to show that Φ_J^* is the connected interval between the two. Note that zero payoff can be achieved by

$$\mu_\circ := \{q_\circ^1, \ldots, q_\circ^K, t_\circ\} \text{ with } q_\circ^k(\mathbf{x}) = t_\circ(\mathbf{x}) = 0 \ \forall (k, \mathbf{x}) \in \mathcal{K} \times \bar{X}^N. \tag{11}$$

For any $\phi \in [0, \bar{\phi}_J]$, there exists a scalar $\theta \in [0,1]$ such that $\phi = \theta \bar{\phi}_J$. Given θ, construct a direct mechanism

$$\mu = (1-\theta)\mu_\circ + \theta \bar{\mu}. \tag{12}$$

Then, $(\mu, \pi) \in Z$ with π based on (9). This is because any convex combination between two BIC direct mechanisms given the same type distribution is also BIC. Accordingly, a seller's ex-ante expected payoff is

$$\Phi_J(\mu, \mu, \pi) = \theta \Phi_J(\bar{\mu}, \bar{\mu}, \pi) = \theta \bar{\phi}_J.$$

Therefore, the set of the seller's feasible ex-ante expected payoffs is the closed connected interval between 0 and $\bar{\phi}_J$. ■

The Bayesian incentive compatibility of a seller's direct mechanism is based on the buyer's interim expected payoff, which depends on the probability that buyers select the seller. Because this selection probability depends on the other sellers' mechanisms as well, Bayesian incentive compatibility generally depends on the other sellers' mechanisms, which makes it difficult to work with BIC direct mechanisms. On the other hand, a dominant strategy incentive compatible (DIC) direct mechanism can be easily used because the dominant strategy incentive compatibility does not depend on buyers' selection strategies. A direct mechanism $\mu = \{q^1, \ldots, q^K, t\}$ is DIC if for all $x, x' \in X$ and all $\mathbf{x}_{-1} = (x_2, \ldots, x_I) \in \bar{X}^{N-1}$

$$\sum_{k \in \mathcal{K}} \left(b^k x + g^k\right) q^k(x, \mathbf{x}_{-1}) + t(x, \mathbf{x}_{-1}) \geq \sum_{k \in \mathcal{K}} \left(b^k x + g^k\right) q^k(x', \mathbf{x}_{-1}) + t(x', \mathbf{x}_{-1}).$$

As shown above, dominant strategy incentive compatibility is based on the buyer's ex-post payoff after she selects a seller and hence a DIC direct mechanism can be defined without reference to other sellers' mechanisms. Let Ω_D be the set of all DIC direct mechanisms. By using the BIC-DIC equivalence in Gershkov, et. al. (2013), we can show that any payoff in Φ_J^* can be supported by a DIC allocation.

Corollary 1 *For any $(\tilde{\mu}, \tilde{\pi}) \in Z$, there exists a DIC allocation $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$ such that*

$$\begin{aligned} U_J(\mu, \mu, \pi, x) &= U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x) \text{ for all } x, \\ \Phi_J(\mu, \mu, \pi) &= \Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}). \end{aligned}$$

Therefore, any ϕ in Φ_J^* can be supported by a DIC allocation, i.e., $(\mu, \pi) \in Z$ with $\mu \in \Omega_D$.

Proof. See Appendix A. ■

Corollary 1 shows that we can focus on a DIC allocation for any feasible ex-ante expected payoff in Φ_J^*. It is also worth noting that Corollary 1 holds whether or not buyers and sellers have limited liability. When they have limited liability, as specified in Section 2, we need to show that $\tilde{\mu}$ also satisfies limited liability given μ that satisfies limited liability.

4 General Competing Mechanisms

Let $\gamma = \{\sigma^1, \ldots, \sigma^K, \tau\}$ be a seller's arbitrary (anonymous) mechanism with a message space $\bar{M} = M \times \{m^\circ\}$ for each buyer.[10] For all $k \in \mathcal{K}$, $\sigma^k : \bar{M}^N \to [0, 1]$ specifies the probability of alternative k as the function of buyers' messages in $\bar{M}^N$. $\tau : \bar{M}^N \to \mathbb{R}_-$ specifies monetary transfer to a buyer as the function of buyers' messages in $\bar{M}^N$. Let Γ be the set of all possible mechanisms with message space $\bar{M}$. Let $\mathcal{P}_\Gamma$ be the set of all (closed) non-empty subsets of Γ. A seller's (anonymous) contract is a mapping $g : H^N \to \mathcal{P}_\Gamma$, where H is a set of messages available for each buyer. Let G_Γ be the set of all possible contracts.

After observing a profile of contracts posted by sellers, each buyer sends a message in H to every seller. A message profile in H^N then determines a subset of mechanisms in $\mathcal{P}_\Gamma$ from which the seller chooses. After seeing a profile of mechanisms, each buyer sends a message in M to only the seller who she selects.

[10] Not participating is equivalent to sending m°.

The general competing mechanism game above may not exactly describe what is happening in on-line markets, but the first-round communication is similar to the search stage where buyers leave various information through html cookies on sellers' web sites they visit as they search. Sellers' web sites are responsive in the sense that they can choose a selling mechanism conditional on what are left in html cookies. The second-round communication can be thought of as the messages that buyers send to the seller whom they select: They are similar to, for example, buyers' bids submitted to a seller's auction site they select after search.

In contrast to the mechanism design with limited commitment (e.g., Bester and Strausz (2000, 2001, 2007)), a principal in our setting (e.g., seller) has full commitment. Note that $\mathcal{P}_\Gamma$ is the set of all closed non-empty subsets of Γ. Therefore, if a principal wants, he can offer a contract that fully specifies a mechanism in Γ conditional on agents' (e.g., buyers') messages in H^N, which leaves nothing for the principal to choose. On the other hand, in mechanism design with limited commitment, a principal cannot restrict the set of his choices conditional on agents' messages.

It is indeed straightforward to show that the standard Revelation Principle holds if there is a single principal in our model in the following sense: For any equilibrium profile of agents' communication strategies and the principal's choice strategy given a contract $g : H^N \to \mathcal{P}_\Gamma$, there exists a corresponding BIC direct mechanism that reproduces the equilibrium allocation with truthful type reporting.

This means that when there is a single principal in the model, it is not necessary to have two rounds of communication and to specify a subset of mechanisms conditional on agents' first-round message from which a principal chooses. We will show that in the model with multiple principals, the set of equilibrium allocations differs.

Given a profile of mechanisms chosen by sellers and buyers' strategies for selecting a seller, buyers' communication strategies generally induce a BIC direct mechanism from a seller's mechanism. It makes it very hard to work with BIC direct mechanisms in our model with multiple sellers because the Bayesian incentive compatibility is based on buyers' interim payoffs, which depends on buyers' selection strategies in a continuation equilibrium. This implies that the Bayesian incentive compatibility of a non-deviator's direct mechanism depends on the deviating seller's mechanism, so the deviator cannot take non-deviators' BIC direct mechanisms as given. On the other hand, it is easy to work with the dominant strategy incentive compatible (DIC) direct mechanisms because it can be defined without reference to buyers' selection strategies or contracts offered by other sellers. For that reason, we are not interested in all possible equilibria but we rather focus on a class of equilibria where non-deviating sellers punish a deviating seller with DIC direct mechanisms.

For this purpose, we only consider a class of contracts that specify a (closed) non-empty subset of DIC direct mechanisms as agents' first-round messages. Let $\mathcal{P}_D$ be the set of all (closed) non-empty subsets of Ω_D. Let G_D be the set of all possible contracts $g : H^N \to \mathcal{P}_D$. We study the general competing mechanism game relative to G_D where each principal simultaneously offers a contract from G_D.[11] After characterizing the set of equilibrium allocations in this section, we show that in Section 5, a seller's choice of a contract $g : H^N \to \mathcal{P}_D$ in an equilibrium of the game relative to G_D is fully optimal for him even if he can choose any other contract in G_Γ.

Our solution concept for an equilibrium of a general competing mechanism game relative to G_D is a symmetric pure-strategy perfect Bayesian equilibrium in which (i) sellers use the same equilibrium *pure* strategies and (ii) buyers play the (symmetric) continuation equilibrium of selecting a seller that is best for a deviating seller upon his deviation. Condition (ii) is a kind of robustness imposed in our solution concept. It means that once the game reaches an equilibrium, a principal has no incentive to deviate to an alternative contract even if he is most optimistic about how likely his contract attracts buyers upon deviation.

From now on, we simply call it an equilibrium unless specified. The first requirement implies that sellers use the same equilibrium strategy on the equilibrium path and that non-deviating sellers use the same strategy to punish a deviating seller. The second requirement means that there is no continuation equilibrium of selecting a seller where a seller gains upon his deviation.

We are interested in the set of a seller's ex-ante expected payoffs that can be supported in an equilibrium of a general competing mechanism game relative to G_D. We first construct a selection behavior when one seller's DIC direct mechanism is μ'

[11] In the case of complete information, the general competing mechanism game described above is the same as one considered in Szentes (2010), where a seller's contract determines a set of actions conditional on buyers' messages and the seller chooses an action from the set. The reason is that the set of all DIC direct mechanisms is the set of all actions in the case of complete information where each buyer's type is drawn from a degenerate probability distribution.

and the other sellers all use a DIC direct mechanism μ. Let us define $\underline{\phi}_J$ as follows:

$$\underline{\phi}_J := \inf_{\mu \in \Omega_D} \left[\sup_{\mu' \in \Omega_D} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \tag{13}$$

Note that the supremum is taken over $\pi(\mu',\mu) \in \Pi(\mu',\mu)$. The idea is that there should not be a continuation-equilibrium strategy of selecting a seller upon his deviation that makes him strictly better off. This is the second requirement of our solution concept.

Note that $\underline{\phi}_J$ may take the form of the following expression:

$$\min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \tag{14}$$

However, (14) may or may not exist. Let Φ^e_J be the set of a seller's ex-ante expected payoffs that can be supported in an equilibrium of a competing mechanism game relative to G_D.

Theorem 1 *If (14) does not exist,* $\Phi^e_J = (\underline{\phi}_J, \bar{\phi}_J]$. *If it does,* $\Phi^e_J := [\underline{\phi}_J, \bar{\phi}_J]$

Proof. We first prove that a DIC allocation, (μ^*, π^*) with $\mu^* \in \Omega_D$, can be supported in an equilibrium of a general competing mechanism game relative to G_D if and only if

$$\Phi_J(\mu^*, \mu^*, \pi^*) > \underline{\phi}_J \tag{15}$$

when (14) does not exist (weak inequality for (15) when (14) exists).

Consider the case where (14) does not exist. Let us prove the "only if" part by contradiction. Suppose that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D but a seller's equilibrium ex-ante expected payoff does not satisfy (15). Because $\underline{\phi}_J$ cannot be reached, it implies that

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \underline{\phi}_J = \inf_{\mu \in \Omega_D} \left[\sup_{\mu' \in \Omega_D} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right]. \tag{16}$$

A seller can deviate to a contract $\tilde{g}$ that leaves the whole set of DIC direct mechanisms Ω_D with him regardless of buyers' messages in H^N. Given a DIC direct mechanism μ that each non-deviating seller chooses in a continuation equilibrium upon a seller's deviation to such a contract $\tilde{g}$, the deviator can then choose μ'' from Ω_D and $\pi''(\mu'', \mu) \in \Pi(\mu'', \mu)$ such that

$$\left|\Phi_J(\mu'', \mu, \pi'') - \underline{\phi}_J\right| < \epsilon. \tag{17}$$

Combining (16) and (17) yields

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \Phi_J(\mu'', \mu, \pi'').$$

This implies that there exists a continuation equilibrium where a seller gains upon deviation. This contradicts that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D. Therefore, (15) must be satisfied.

If (14) exists, then the expression on the right hand side of the equality in (16) is (14). Given a DIC direct mechanism μ that each non-deviating seller chooses in a continuation equilibrium upon a seller's deviation to such a contract $\tilde{g}$, the deviator can then choose μ' from Ω_D and $\pi'(\mu', \mu) \in \Pi(\mu', \mu)$ such that

$$\mu' \in \arg\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right).$$

Because such a deviation must not be profitable in equilibrium, we have that

$$\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \leq \Phi_J(\mu^*, \mu^*, \pi^*) \tag{18}$$

Since the expression on the right hand side of the equality in (16) is (14), combining (16) and (18) yields

$$\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) < \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu', \mu) \in \Pi(\mu', \mu)} \Phi_J(\mu', \mu, \pi) \right) \right],$$

which cannot be true. Therefore, (15) must be satisfied with weak inequality.

Now we prove the "if" part, that is, if (μ^*, π^*) satisfies (15), then it can be supported in a pure-strategy equilibrium of a competing mechanism game relative to G. First, consider the case where (14) does not exist. Divide the message space H of g to two non-empty disjoint subsets, B and B^c such that $B \cup B^c = H$. Let h_i denote buyer i's message in H. All sellers post a contract g^* such that

$$g^*(h_1,\ldots,h_n) := \begin{cases} \mu_p \in \Omega_D & \text{if } |\{i : h_i \in B\}| > N/2, \\ \mu^* \in \Omega_D & \text{otherwise,} \end{cases} \tag{19}$$

where μ_p satisfies

$$\sup_{\mu' \in \Omega_D} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu_p, \pi) \right) < \Phi_J(\mu^*, \mu^*, \pi^*) \tag{20}$$

If *all* sellers post g^*, then all buyers send messages in B^c and g^* assigns μ^*. Agents select sellers according to $\pi^*(\mu^*, \mu^*)$ and send their true types to a seller that they select. If one seller deviates to g, then all buyers send messages in B to the non-deviating seller. The non-deviator's contract g^* then assigns μ_p. Because (μ^*, π^*) satisfies (15), the deviating seller cannot gain by choosing any DIC direct mechanism, given each non-deviator's DIC direct mechanism μ_p, satisfying (20).

Now consider the case where (14) exists. We can prove that a DIC allocation, (μ^*, π^*) with $\mu^* \in \Omega_D$, can be supported in an equilibrium of a general competing mechanism game relative to G_D if and only if (15) holds with weak inequality. For the proof of the "only if" part, suppose that (μ^*, π^*) is supported in an equilibrium of a general competing mechanism game relative to G_D but a seller's equilibrium ex-ante expected payoff does not satisfy (15) with weak inequality:

$$\Phi_J(\mu^*, \mu^*, \pi^*) < \underline{\phi}_J = \min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right]$$

If (14) exists, we can choose μ_p that satisfies

$$\mu_p \in \arg\min_{\mu \in \Omega_D} \left[\max_{\mu' \in \Omega_D} \left(\max_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right) \right].$$

Then, $\underline{\phi}_J$ can be supported as a seller's equilibrium ex-ante expected payoff.

What we proved above is that any DIC allocation that generates a seller's ex-ante expected payoff no less than $\underline{\phi}_J$ can be supported in a pure-strategy equilibrium. Clearly, $\bar{\phi}_J$ is the maximum. We need to prove that any payoff between $\underline{\phi}_J$ and $\bar{\phi}_J$ is supportable in an equilibrium of a competing mechanism game relative to G to complete the proof. Note that $\Phi^e_J = [\underline{\phi}_J, \bar{\phi}_J] \subset \Phi^*_J$ because $\underline{\phi}_J \geq 0$. According to

Corollary 1, any ϕ in Φ^e_J can be induced by some DIC allocation, $(\mu^*, \pi^*) \in Z$ with $\mu^* \in \Omega_D$. Therefore, any ϕ in Φ^e_J can be supported in an equilibrium where every seller posts a contract g^* that assigns the corresponding μ^* when no seller deviates. ■

In a general competing mechanism game relative to G_D described above, a seller can deviate to a contract that always specifies the whole set of DIC direct mechanism Ω_D regardless of buyers' first-round messages. This makes it possible for a deviating seller to choose a DIC direct mechanism from Ω_D that maximizes his ex-ante expected payoff given a profile of DIC direct mechanisms that he believes non-deviators would choose. Therefore, the lower bound of a seller's equilibrium ex-ante expected payoff in a general competing mechanism game relative to G_D rises to the minmax value in terms of DIC direct mechanisms.

On the other hand, if a contract is restricted to specify a single DIC direct mechanism conditional on buyers' first-round messages, non-deviators can choose their DIC direct mechanisms conditional on the deviating seller's mechanism. This is why the lower bound of a seller's ex-ante expected payoff is the maxmin value, which is generally no greater than the minmax value.

This result holds more generally. The lower bound of a principal's equilibrium payoff in the game relative to G_Γ is the minmax value over mechanisms in Γ. On the other hand, the lower bound of a principal's equilibrium payoff is the maxmin value over mechanisms in Γ when the game allows a principal to offer only a contract that always specifies a single mechanism in Γ conditional on buyers' first-round messages (See Han (2021), Han and Xiong (2021) for related results).[12]

It is worth noting that in order to lower a deviating seller's payoff to $\underline{\phi}_J$, the messages in non-deviating sellers' equilibrium contracts can be as simple as binary (i.e., whether or not there exists a deviating seller), as (19) shows. For this, let $H = \{0, 1\}$, $B = \{1\}$, and $B^c = \{0\}$ so that $B \cup B^c = H$. The message 1 implies that the competing seller deviated. The message 0 means that no sellers deviated. Binary messages can be easily adopted in practice (e.g., on-line markets). Let us call g^* with $H = \{0, 1\}$ a deviation-reporting contract. If sellers are heterogenous, then non-deviating sellers need to know the identity of a deviating seller. In this case, H should be $\{0, 1, \ldots, J\}$ where a non-zero number represents the identity of a deviating

[12] Han (2021) and Han and Xiong (2021) consider more general environments that a player's payoff (agent or principal) depends on all agents' types and all principal's actions.

seller, with zero taken to mean that there are no deviations by any competing seller.

5 Equilibrium Allocations with Strategy-proof Punishment

One may wonder if an equilibrium DIC allocation in a general competing mechanism game relative to G_D is also an equilibrium allocation in a general competing mechanism game relative to G_Γ where G_Γ is the set of all possible contracts that specify a menu of mechanisms in Γ conditional on buyers' messages in H^N.

Let us fix an equilibrium DIC allocation (μ^*, π^*) in a general competing mechanism game relative to G_D and let $g^* \in G_D$ be the equilibrium contract that supports (μ^*, π^*). The question we address in this section is whether it is fully optimal for a seller to choose g^* when he can choose any contract in G_Γ. If it is, the equilibrium in the game relative to G_D is said to be a *robust* equilibrium. We assume that Γ is large enough that the set of all direct mechanisms, including all DIC direct mechanisms, can be embedded into Γ. Given our assumption on Γ, it is also clear that $g^* \in G_\Gamma$ with a slight abuse of notation (More precisely, there exists a contract g' in G_Γ such that g^* is homeomorphic to it).

When a seller deviates, each non-deviating seller's contract g^* will choose a DIC direct mechanism μ_p given that buyers all report that his competitor deviated. Suppose that a deviating seller posts a contract g that assigns a menu of mechanisms in Γ and chooses a mechanism γ from the menu that is assigned. Then, an array of mechanisms (γ, μ_p) defines a subgame played by buyers.[13] We can fix truthful type reporting to each non-deviating seller because μ_p is DIC. Then, a continuation equilibrium of the subgame defined by (γ, μ_p) is characterized by (a) the buyer's strategy of communicating with the deviating seller, $c(\gamma, \mu_p) : X \rightarrow \Delta(M)$ upon selecting him and (b) her strategy of selecting him, $\pi(\gamma, \mu_p) : X \rightarrow [0, 1]$.

The buyer's communication strategy $c(\gamma, \mu_p)$ induces a direct mechanism $\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ from $\gamma = \{\sigma^1, \ldots, \sigma^K, \tau\}$. Let I denote the number of buyers who select the deviating seller. Then, for every $I \leq N$ and every $(x_1, \ldots, x_I) \in X^I$,

[13] (γ, μ_p) means that the deviating seller's mechanism is γ, whereas every non-deviating principal's seller is μ_p.

$\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ is defined as, for all $k \in \mathcal{K}$,

$$q^k_{c,\mu_p}(x_1, \ldots, x_I, \mathbf{x}^\circ_{-I}) = \int_M \cdots \int_M \sigma^k(m_1, \ldots, m_I, \mathbf{m}^\circ_{-I}) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

$$t_{c,\mu_p}(x_1, \ldots, x_I, \mathbf{x}^\circ_{-I}) = \int_M \cdots \int_M \tau^k(m_1, \ldots, m_I, \mathbf{m}^\circ_{-I}) dc(\gamma, \mu_p)(x_1) \times \cdots \times dc(\gamma, \mu_p)(x_I),$$

where $\mathbf{x}^\circ_{-I} = (x^\circ, \ldots, x^\circ)$ and $\mathbf{m}^\circ_{-I} = (m^\circ, \ldots, m^\circ)$.

The buyer's selection strategy $\pi(\gamma, \mu_p)$ induces the probability $z(\pi(\gamma, \mu_p))(x)$ that she either has her type below x as a participant of γ or selects the other sellers whose mechanism is μ_p, similar to (3):

$$z(\pi(\gamma, \mu_p))(x) = 1 - \int_x^{\bar{x}} \pi(\gamma, \mu_p)(s) dF. \tag{21}$$

Let $\mathcal{O}$ be the set of all optimal strategies (c, π) for communicating with the deviating seller and selecting a seller in a continuation equilibrium of the subgame defined by (γ, μ_p) for all $\gamma \in \Gamma$. Following (4) and (5), we can derive the reduced-form direct mechanism, $\{Q^1_{\pi,c,\mu_p}, \ldots, Q^K_{\pi,c,\mu_p}, T_{\pi,c,\mu_p}\}$ from $\mu_{c,\mu_p}(\gamma) = \{q^1_{c,\mu_p}, \ldots, q^K_{c,\mu_p}, t_{c,\mu_p}\}$ and $z(\pi(\gamma, \mu_p))$. Then, it is straightforward to show that $\mu_{c,\mu_p}(\gamma)$ is Bayesian incentive compatible (BIC) for any $(c, \pi) \in \mathcal{O}$: For all $x, x' \in X$,

$$\sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k_{\pi,c,\mu_p}(x) + T_{\pi,c,\mu_p}(x) \geq \sum_{k \in \mathcal{K}} (b^k x + g^k) Q^k_{\pi,c,\mu_p}(x') + T_{\pi,c,\mu_p}(x').$$

Note that the Bayesian incentive compatibility of $\mu_{c,\mu_p}(\gamma)$ depends on the mechanism μ_p chosen by the other sellers because the buyer's communication and selection strategies depend on mechanisms chosen by both sellers.

Following (6) and (8), we can then use the reduced-form direct mechanism to derive (i) the interim expected payoff, denoted by $U_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi, x)$, for the buyer of type x associated with selecting the deviating seller with γ and (ii) the ex-ante expected payoff, denoted by $\Phi_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi)$, for the deviating seller with γ. Our next theorem is the key result in this section.

Theorem 2 *Any equilibrium (DIC) allocation in a competing mechanism game relative to G_D is also supported in an equilibrium of a general competing mechanism game relative to G_Γ.*

We prove Theorem 2 by two steps. We start with some basics. Let Ω be the set of all direct mechanisms. Let $\mathcal{O}_c$ be the projection of $\mathcal{O}$ onto the space of the buyer's communication strategies. Then, let $\mathcal{M}(\mu_p)$ be the set of *all BIC direct mechanisms* that can be induced from all mechanisms in Γ that a deviating seller might choose in a continuation equilibrium given the non-deviating seller's DIC direct mechanism $\mu_p \in \Omega_D$:

$$\mathcal{M}(\mu_p) := \left\{ \mu_{c,\mu_p}(\gamma) \in \Omega : \forall c \in \mathcal{O}_c,\ \forall \gamma \in \Gamma \right\}. \tag{22}$$

Let $\Pi(\mu, \mu_p)$ be the set of all optimal strategies of selecting a seller in a continuation equilibrium where the deviating seller chooses an incentive compatible direct mechanism μ directly from $\mathcal{M}(\mu_p)$. We can then establish the following lemma.

Lemma 2 *Given $\mu_p \in \Omega_D$, the following equality holds:*

$$\sup_{\gamma \in \Gamma} \left(\sup_{(c,\pi) \in \mathcal{O}} \Phi_J(\mu_{c,\mu_p}(\gamma), \mu_p, \pi) \right) = \sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu,\mu_p) \in \Pi(\mu,\mu_p)} \Phi_J(\mu, \mu_p, \pi) \right). \tag{23}$$

Proof. Given μ_p that each non-deviating seller chooses, suppose that the deviating seller's mechanism is $\gamma \in \Gamma$. Given (γ, μ_p), let each buyer communicate with the deviating seller upon selecting him and select a seller according to $c(\gamma, \mu_p)$ and $\pi(\gamma, \mu_p)$ for some $(c, \pi) \in \mathcal{O}$. Since $\mathcal{O}$ is the set of all possible communication and selection strategies in a continuation equilibrium upon a seller's deviation to a mechanism in Γ, $\mu_{c,\mu_p}(\gamma) \in \mathcal{M}(\mu_p)$ and further it is a continuation equilibrium that each buyer uses $\pi(\mu_{c,\mu_p}(\gamma), \mu_p) = \pi(\gamma, \mu_p)$ to select a seller and truthfully report her type to the seller who she selects when the deviating seller directly chooses $\mu_{c,\mu_p}(\gamma)$. This implies (23) because $\mathcal{M}(\mu_p)$ is a subset of Ω, which is embedded into Γ. ■

Lemma 2 implies that an equilibrium DIC allocation (μ^*, π^*) in a general competing mechanism game relative to G_D is also an equilibrium of a general competing mechanism game relative to G_Γ if

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu,\mu_p) \in \Pi(\mu,\mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \tag{24}$$

We can then use the BIC-DIC equivalence to establish (24).

Corollary 2 *Suppose that a deviating seller offers* $\tilde{\mu} \in \mathcal{M}(\mu_p)$, *whereas each non-deviating seller uses* μ_p *to punish the deviator and that it is a continuation equilibrium that buyers chooses the deviator according to* $\tilde{\pi}(\tilde{\mu}, \mu_p)$ *given* $(\tilde{\mu}, \mu_p)$. *Then, there exists a DIC direct mechanism* $\mu \in \Omega_D$ *such that given* (μ, μ_p), *it is a continuation equilibrium that buyers choose the deviator according to* $\pi(\mu, \mu_p) = \tilde{\pi}(\tilde{\mu}, \mu_p)$ *and*

$$U_J(\mu, \mu_p, \pi, x) = U_J(\tilde{\mu}, \mu_p, \tilde{\pi}, x) \ , \forall x \in X \tag{25}$$

$$\Phi_J(\mu, \mu_p, \pi) = \Phi_J(\tilde{\mu}, \mu_p, \tilde{\pi}). \tag{26}$$

Proof. The proof is almost the same as the proof of Corollary 1. The only difference is that, with respect to the deviating seller's point of view, the probability that either a buyer's type is less than x or she selects a non-deviator is calculated based on $\pi(\mu, \mu_p)$ and $\tilde{\pi}(\tilde{\mu}, \mu_p)$, that is, $z(\pi(\mu, \mu_p))(x)$ and $z(\tilde{\pi}(\tilde{\mu}, \mu_p))(x)$. ∎

To complete the proof of Theorem 2, first note that (15) and (20) imply that a equilibrium DIC allocation (μ^*, π^*) in a competing mechanism game relative to G_D satisfies

$$\Phi_J(\mu^*, \mu^*, \pi^*) \geq \sup_{\mu \in \Omega_D} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right)$$

Because $\Omega_D \subset \mathcal{M}(\mu_p)$, it is clear that

$$\sup_{\mu \in \mathcal{M}(\mu_p)} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \geq \sup_{\mu \in \Omega_D} \left(\sup_{\pi(\mu, \mu_p) \in \Pi(\mu, \mu_p)} \Phi_J(\mu, \mu_p, \pi) \right) \tag{27}$$

In Corollary 2, (26) means that (27) holds with equality and hence (24) is satisfied for any equilibrium allocation in a general competing mechanism game relative to G_D. This implies that any equilibrium allocation in a general competing mechanism game relative to G_D is also supported in an equilibrium of a general competing mechanism game relative to G_Γ.

Applying the BIC-DIC equivalence in Corollary 1 to any BIC equilibrium allocation, we can derive the following implications from Theorem 2.

Theorem 3 Φ_J^e *is the complete set of a seller's ex-ante payoffs associated with all equilibrium allocations that are supportable with strategy-proof punishment (i.e., punishment carried out by a DIC direct mechanism) in any general competing mechanism game* G_Γ.

5.1 Remarks on BIC-DIC equivalence

Even when the BIC-DIC equivalence does not hold, we can still characterize the set of "robust" equilibrium allocations with strategy-proof punishment, using incentive compatible direct mechanisms (that is, allocations with strategy-proof punishment that survive as equilibrium allocations even when a seller can offer any contract in G_Γ). In this case, players' payoff functions can be quite general without imposing the linearity in (1) and (2). A (robust) equilibrium allocation is BIC and the lower bound of a seller's equilibrium ex-ante expected payoff is

$$\inf_{\mu\in\Omega_D} \sup_{\mu'\in\mathcal{M}(\mu)} \left(\sup_{\pi(\mu',\mu)\in\Pi(\mu',\mu)} \Phi_J(\mu',\mu,\pi) \right), \tag{28}$$

where $\mathcal{M}(\mu)$ is the set of all BIC direct mechanisms available for a seller conditional on the other sellers' DIC direct mechanisms $\mu \in \Omega_D$. The upper bound is a seller's payoff associated with a (robust) equilibrium allocation that is derived by a joint profit maximization.

The BIC-DIC equivalence given the linearity in (1) and (2) only ensures that a seller can focus on DIC direct mechanisms without loss of generality when he deviates instead of BIC direct mechanisms, given non-deviating sellers' contracts that use DIC direct mechanisms to punish him. Therefore, it implies two things. First, we can replace $\mathcal{M}(\mu)$ below the first supremum in (28) with Ω_D as shown in (13). Second, given the profile of the other sellers' contracts, the equilibrium DIC direct mechanism implemented by each principal provides *full optimality* for him among all possible mechanisms.

5.2 Heterogenous buyers and sellers

Given ex-ante homogenous buyers and sellers, we focus on a symmetric equilibrium where sellers use the same pure strategies. This is mainly for notational simplicity. We can relax the homogeneity assumption with heterogenous payoff parameters, and allow for different numbers of action alternatives for sellers, and different probability distributions for buyers.

For endogenous frictions, we still want to impose symmetry in buyers' selection strategies in that they select sellers with equal probability if their mechanisms are the same. Given this symmetric selection behavior, the main results go through in the sense that the greatest lower bound of a principal's robust equilibrium payoff supportable with DIC punishment is his minmax value over DIC direct mechanisms.[14] The upper bound is not well defined since it is not clear how to set up the joint profit maximization for heterogenous sellers.

6 Applications

Each buyer has a unit demand for a product. If she consumes the product and pays p, her utility is $x - p$, where x is her valuation that follows a probability distribution F over $X = [0, 1]$. There are N buyers who are looking for the product. Each buyer's reservation utility is zero. There are J sellers. Sellers and buyers are all risk neutral.

6.1 Competing prices

We consider the case where sellers can produce homogeneous products at a constant marginal cost, normalized to zero, without capacity constraint. Each seller's reservation profit is zero.

Theorem 4 *Suppose that the hazard rate, $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x.*

1. *The set of a seller's ex-ante expected profit that can be supported in a robust equilibrium with DIC punishment of a competing mechanism game relative to G_Γ is $\Phi^e_J = \Phi^*_J = [0, \overline{\phi}_J]$ with*

$$\overline{\phi}_J = \frac{N}{J} x^*(1 - F(x^*)), \tag{29}$$

   where x^* is uniquely defined as the value of x satisfying $x - \frac{1-F(x)}{f(x)} = 0$.

2. *Any $\phi \in \Phi^*_J = [0, \overline{\phi}_J]$ can be supported in a robust equilibrium where each seller posts a deviation-reporting contract g^* that offers a single price contingent on buyer's messages $(h_1, \ldots, h_N) \in H^N$ with $H = \{0, 1\}$ such that*

$$g^*(h_1, \ldots, h_N) := \begin{cases} 0 & \text{if } |\{i : h_i = 1\}| > N/2, \\ x(\phi) & \text{otherwise,} \end{cases} \tag{30}$$

   where $x(\phi)$ satisfies $\phi = \frac{N}{J} x(1 - F(x))$.

[14] The results are available upon request. The notation is considerably heavier.

Proof. See Appendix B. ■

In the seller's joint profit maximization, all sellers post an identical direct mechanism $\mu = \{q, p\}$ such that mappings $q : X^N \to [0, 1]$ and $p : X^N \to \mathbb{R}_+$ specify the probability that a buyer receives the product and her payment to the seller respectively (Note that $X = [0, 1]$ so that, if a buyer does not choose the seller, her type message is regarded as zero and $q(0, \mathbf{x}_{-i}) = p(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$). Because every seller's mechanism is identical, a buyer of any type x selects a seller with $\pi(\mu, \mu)(x) = 1/J$.[15] For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ. One can then derive a reduced-form direct mechanism $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ similar to (4) and (5), based on $z(\pi)(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ in (9).

Because a seller can produce each unit at zero constant cost and each buyer's valuation is i.i.d., the seller's joint profit maximization problem is to find $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_+$ that maximize

$$N \int_0^1 P(x) dz(\pi)(x) \tag{31}$$

subject to the incentive compatibility and individual rationality conditions. We can apply the standard monopolist's analysis (e.g., p. 265 in Fudenberg and Tirole 1991) to find out a solution for joint profit maximization. Given (9), we can show that the virtual valuation of a buyer with valuation x is

$$x - \frac{1 - z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1 - F(x)}{f(x)}.$$

Given the monotone hazard rate on $x \in [0, 1]$, it is then clear that the product must be sold only to a buyer whose virtual valuation is equal to zero or higher with probability one. A buyer's payment is then

$$P(x) = xQ(x) - \int_0^x Q(s) ds = \begin{cases} x^* & \text{if } x \geq x^*, \\ 0 & \text{otherwise.} \end{cases} \tag{32}$$

This induces the seller's ex-ante expected profit in the joint profit maximization as (29). Furthermore, any feasible ex-ante profit for a seller $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$ can be

[15]If a buyer does not choose any seller, it is equivalent to choosing a seller with equal probability and sending the zero type message.

expressed as $\phi = \frac{N}{J}x(\phi)(1 - F(x(\phi)))$ for some $x(\phi) \in [0,1]$ as shown in Appendix B. Therefore, any feasible ex-ante profit ϕ for a seller can be supported in a robust equilibrium where each seller's deviation-reporting contract assigns either zero (constant marginal cost) or $x(\phi)$, depending on buyers' messages on whether or not a competing seller deviates.

6.1.1 Fixed entry cost

Given the number of buyers N, we have assumed that there is a fixed number of sellers in the market. However, we can endogenize the number of sellers in the market. Suppose that a seller has to incur a fixed cost, $C > 0$, to enter the market.

Note that a seller's *gross* ex-ante expected profit $\phi \in \Phi_J^*$ is an equal split of $Nx(1-F(x))$ as shown in (52) in Appendix B. Assume that $\frac{N}{2}x^*(1-F(x^*))-C \geq 0$.[16] Recall that $\frac{N}{J}x(1-F(x))$ is maximized at $x = x^*$ by excluding buyers whose valuation is less than x^*. Given the monotone hazard rate on x, a sellers' *ex-ante expected net profit* $\frac{N}{J}x(1 - F(x)) - C$ is increasing in x before $x = x^*$ and decreasing in x after. Furthermore, it is equal to $-C$ at $x = 0$ or 1.

We look for an equilibrium in which there is at least two sellers in the market. Define X_N^* as

$$X_N^* := \left\{ x \in X : \frac{N}{2}x(1 - F(x)) - C \geq 0 \right\}.$$

X_N^* is a closed interval because $\frac{N}{2}x(1-F(x))-C$ is increasing in x before $x = x^*$ and decreasing in x after. For any $x \in X_N^*$, let $J_N^*(x)$ be the maximum natural number that satisfies

$$\frac{N}{J_N^*(x)}x(1 - F(x)) - C \geq 0$$

Proposition 1 *Given the number of buyers N,*

1. *any price in X_N^* is supported in a robust equilibrium with strategy-proof punishment and a fixed cost of entry C.*

2. *for any $x \in X_N^*$, $J_N^*(x)$ is the number of sellers in the market in a robust equilibrium with strategy-proof punishment and a fixed cost C*

Proof. Because we look for an equilibrium in which there is at least two sellers in the market, it is necessary for the equilibrium price to be in X_N^*. Suppose that $x \in X_N^*$

[16] This assumption implies that there will be at least two sellers in the market.

is the price that sellers believes would be prevailing in equilibrium. Given $x \in X_N^*$, sellers would keep entering until their ex-ante expected net profit is non-positive. Therefore, the equilibrium number of sellers is $J_N^*(x)$. A price $x \in X_N^*$ is implemented by sellers in the market posting the deviation-reporting contract g^* that assigns x when no deviation is reported by every buyer, 0 otherwise. ■

For any given $x \in X_N^*$, the equilibrium profit can strictly be positive with a finite number of buyers. To see this, let $s_N^*(x)$ be the ratio of the number of buyers to the number of sellers that satisfies

$$s_N^*(x)x(1 - F(x)) - C = 0$$

Then, $J_N^*(x)$ is the largest natural number such that $N/J_N^*(x)$ does not exceed $s_N^*(x)$. With a finite number of buyers, it is possible that $N/J_N^*(x)$ is strictly less than $s_N^*(x)$. In this case, a seller's ex-ante expected profit is strictly positive. However, as N approaches infinity, $N/J_N^*(x)$ approaches $s_N^*(x)$, which makes every seller's ex-ante expected net profit equal to zero.

Let $s_\infty^*(x)$ be the ratio of the number of buyers to sellers in the market as N approaches infinity. Now we define

$$X_\infty^* := \{x \in X : \exists s_\infty^*(x) > 0 \text{ s.t. } s_\infty^*(x)x(1 - F(x)) - C = 0\}.$$

Proposition 2 *As $N \to \infty$, any price in X_∞^* is supported in a robust equilibrium with strategy-proof punishment and a fixed cost of entry C, the equilibrium ratio of the number of buyers to the number of sellers approaches $s_\infty^*(x)$ and a seller's ex-ante expected equilibrium profit approaches zero for all $x \in X_\infty^*$.*

Proof. Given the number of buyers N and $x \in X_N^*$, it is clear that

$$\left| s_N^*(x) - \frac{N}{J_N^*(x)} \right| \to 0 \text{ as } N \to \infty.$$

Further, $s_N^*(x)$ becomes $s_\infty^*(x)$ as $N \to \infty$. ■

The multiplicity of equilibria even in a large market stems from the multiplicity of the equilibrium price. Given sellers' belief on the equilibrium price x in X_∞^*, they keep entering the market as long as their ex-ante expected net profit associated with x is non-negative. Therefore, the equilibrium ratio of the number of buyers to the number of sellers is equal to $s^*_\infty(x)$ and a seller's equilibrium ex-ante expected net profit is always equal to zero in a large market regardless of the equilibrium price, whereas a seller's equilibrium ex-ante expected net profit can be positive in a finite market. In both markets, the equilibrium ratio of the number of buyers to the number of sellers is the lowest at the monopoly price x^* and it increases as the equilibrium price moves farther away from x^* in either direction. The large market result holds even when each seller faces a capacity constraint. We show this in the auction environment examined in the next subsection.

6.2 Competing auctions

In Section 6.1, each seller has no capacity constraint. What if each seller can produce at most a single unit? The literature has studied competing auctions where sellers are restricted to choose their reserve prices in the (second price) auction (Burguet and Sákovics 1999, Peters 1997, Peters and Severinov 1997, Virag 2010). It shows that there is a pure-strategy equilibrium of the competing auction game in a *large market* (Peters 1997, Peters and Severinov 1997) where the number of buyers and sellers goes to infinity given a fixed ratio of buyers to sellers and that it is robust in that no seller can gain by deviating to any arbitrary mechanism (Han 2015). In this equilibrium, every seller sets a reserve price equal to his cost of producing the product. While the literature has found one robust equilibrium, it is not yet known if there are additional robust equilibria.

Theorem 5 below identifies a seller's jointly maximized ex-ante expected profit, $\overline{\phi}_J$.

Theorem 5 *Suppose that the hazard rate $h(x) = \frac{f(x)}{1-F(x)}$ is non-decreasing in x. Then $\overline{\phi}_J$ is reached when all sellers offer an auction with reserve price x^* such that*

$$x^* - \frac{1-F(x^*)}{f(x^*)} = 0 \tag{33}$$

and it is

$$\overline{\phi}_J = \frac{N}{J}\int_{x^*}^{1}\left(x - \frac{1-F(x)}{f(x)}\right)\left(1 - \frac{1-F(x)}{J}\right)^{N-1} f(x)dx. \tag{34}$$

Let s be the ratio of the number of buyers to the number of sellers, i.e., $s = \frac{N}{J}$. In a *large market, it becomes*

$$\overline{\phi}_\infty := \lim_{J\to\infty} \overline{\phi}_J = s\int_{x^*}^{1} \left[\left(x - \frac{1-F(x)}{f(x)}\right) e^{-s(1-F(x))}\right] f(x)dx. \qquad (35)$$

Proof. See Appendix C. ■

$\overline{\phi}_J$ is a seller's ex-ante expected profit in the joint profit maximization where every seller uses an identical selling mechanism. Therefore, each buyer with valuation x selects a seller with equal probability $\pi(\mu,\mu))(x) = 1/J$ and hence the probability that a buyer whose valuation is less than x or she selects another seller is given by

$$z(\pi(\mu,\mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x.$$

Then, a buyer's virtual valuation is

$$x - \frac{1 - z(\pi)(x)}{z'(\pi)(x)} = x - \frac{1-F(x)}{f(x)}.$$

Given the monotone hazard rate, the virtual valuation is increasing and the optimal mechanism takes the form of an auction with a reserve price: The object goes to the highest valuation buyer if it is sold at all. It is sold if and only if the highest valuation among participating buyers is no less than x^*. Appendix C shows that (34) and (35) are a seller's jointly maximized ex-ante expected profits in finite and large markets respectively.

The lower bound of a seller's robust equilibrium ex-ante expected profit $\underline{\phi}_J$ is defined as his minmax value over all possible DIC direct mechanisms according to its definition in (13). However, it is difficult to derive $\underline{\phi}_J$ when there are both limited liability and capacity constraint as in the auction environment. Here, we instead focus on the set of a seller's ex-ante expected profits that can be supported in a robust equilibrium where non-deviating sellers punish the deviator by changing the reserve price in their auctions. The lower bound of a seller's ex-ante expected profits is then expressed in terms of the minmax value with respect to reserve prices in the auctions.

Let $\mu(x)$ denote an auction with reserve price x. Then, $\Phi_J(\mu(x'), \mu(x), \pi)$ denotes a seller's expected profit when he posts an auction with reserve price x' given that $J-1$ sellers all post auctions with reserve price x and buyers select a seller according to their selection strategy π.

Peters (1997) shows that in a large market, a deviating seller cannot do better with any arbitrary direct mechanism than he does with an auction, given any distribution of reserve prices of auctions chosen by non-deviating sellers. We can apply the result in Han (2015) to show that in a large market, a deviating seller cannot do better with any arbitrary mechanism than he does with a direct mechanism, given any distribution of reserve prices of auctions chosen by non-deviating sellers. Therefore, we have that, for all $x \in [0,1]$,[17]

$$\lim_{J\to\infty}\left[\sup_{\gamma\in\Gamma}\left(\sup_{(c,\pi)\in\mathcal{O}}\Phi_J(\gamma,\mu(x),c,\pi)\right)\right]=\lim_{J\to\infty}\left[\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x),\pi)\right]. \tag{36}$$

This implies that a deviating seller only needs to consider an auction with reserve price even if he can offer any arbitrary selling mechanism. Importantly, (36) holds independent of the non-decreasing property of the buyer's virtual valuation. This is the key to the robustness in a large market.[18] Let

$$\underline{\phi}_{\infty}:=\lim_{J\to\infty}\left[\min_{x\in[0,1]}\left(\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x),\pi)\right)\right].$$

Proposition 3 $\tilde{\Phi}_{\infty}=\left[\underline{\phi}_{\infty},\overline{\phi}^{\infty}\right]$ *is the range of a seller's ex-ante expected profits that can be supported in a robust equilibrium of a competing mechanism game relative to* G_{Γ}*, where non-deviating sellers punish a deviating seller by changing their reserve prices.*

Proof. Let $x^p \in [0,1]$ be a reserve price such that

$$\lim_{J\to\infty}\left[\max_{x'\in[0,1]}\Phi_J(\mu(x'),\mu(x^p),\pi)\right]=\underline{\phi}_{\infty}.$$

[17]For any given array of auctions offered by sellers, there exists a unique selection strategy π in a continuation equilibrium (Peters and Severinov 1997, Peters 1997, Virag 2010). This is why the maximum over the selection strategies is not taken.

[18]For equilibrium characterization and comparative statics, we consider a large market. Nonetheless, it is worthwhile to note that the competing auctions may not have a pure-strategy equilibrium in a market with a finite number of sellers because non-deviators cannot change their reserve prices upon a seller's deviation. In our competing mechanism game, non-deviators lower their reserve price down to zero, attracting a lot more buyers, upon a seller's deviation. This dampens a seller's incentive to deviate even in a market with a finite number of sellers. The analysis based on a finite number of sellers is of its own interest but it is beyond scope of our paper.

Then, we have

$$\lim_{J\to\infty} \Phi_J(\mu(x^p), \mu(x^p), \pi) \le \lim_{J\to\infty} \left[\max_{x'\in[0,1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \underline{\phi}_\infty \tag{37}$$

by the definition of the maximum operator. On the other hand, we have that

$$\lim_{J\to\infty} \Phi_J(\mu(x^*), \mu(x^*), \pi) = \overline{\phi}^\infty \ge \lim_{J\to\infty} \left[\max_{x'\in[0,1]} \Phi_J(\mu(x'), \mu(x^p), \pi) \right] = \underline{\phi}_\infty \tag{38}$$

Furthermore, when every seller's reserve price is the same, a seller's ex-ante expected profit,

$$\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi) = s \int_x^1 \left[\left(x' - \frac{1-F(x')}{f(x')} \right) e^{-s(1-F(x'))} \right] f(x')dx',$$

is continuous in x. Given the continuity of $\lim_{J\to\infty} [\Phi_J(\mu(x), \mu(x), \pi)]$, (37) and (38) imply that there exists at least one x such that

$$\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi) = \underline{\phi}_\infty.$$

Furthermore, given the non-decreasing hazard rate and the definition of x^*,

(a) $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-increasing in x over $[x^*, 1]$ while it reaches its maximum at $x = x^*$ and its minimum at $x = 1$ in the range of $[x^*, 1]$, and

(b) $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is non-decreasing in x over $[0, x^*]$, while it reaches its maximum at $x = x^*$ and its minimum at $x = 0$ in the range of $[0, x^*]$.

Because $\lim_{J\to\infty} \Phi_J(\mu(x), \mu(x), \pi)$ is continuous in x, (a) and (b) imply that there exists at least one $x(\phi)$ such that $\lim_{J\to\infty} \Phi_J(\mu(x(\phi)), \mu(x(\phi)), \pi) = \phi$ for all $\phi \in \tilde{\Phi}_\infty$.

For any given $\phi \in \tilde{\Phi}_\infty$, each seller posts the deviation-reporting contract

$$g^*(h_1, \ldots, h_N) := \begin{cases} \mu(x^p) & \text{if } |\{i : h_i = 1\}| \ge \frac{N}{2} \\ \mu(x(\phi)) & \text{otherwise} \end{cases}. \tag{39}$$

Given this contract, buyers truthfully report whether or not the competing seller deviates to all sellers, and also their true valuation upon selecting a seller. Because $\phi \ge \underline{\phi}_\infty$, (36) implies that a seller cannot gain by deviating to any arbitrary mechanism in a large market. ■

For any ex-ante expected profit level ϕ, we can identify at least one reserve price

$x(\phi)$ that induces each seller's ex-ante expected profit equal to ϕ when every seller offers an auction with $x(\phi)$. It is then straightforward to maintain the profit level ϕ through the deviation-reporting contract that makes the reserve price a function of reports by buyers on whether or not a competing seller deviates, as specified in (39).

6.2.1 Fixed entry cost

Suppose that any seller can enter the market but has to incur a fixed entry cost of C. We can then endogenize the equilibrium ratio of buyers to sellers.

Proposition 4 *Consider a robust equilibrium of a competing mechanism game relative to G_Γ, where non-deviating sellers punish a deviating seller by changing their reserve prices and any seller can freely enter the market at the cost of C. Any reserve price in the following set can be supported in a robust equilibrium with strategy-proof punishment:*

$$X^* := \left\{x \in X : \exists s^*(x) > 0 \ s.t. \ s^*(x) \int_x^1 \left[\left(x' - \frac{1 - F(x')}{f(x')}\right) e^{-s(1-F(x'))}\right] f(x')dx' - C = 0\right\}$$

Given an equilibrium reserve price $x \in X^*$, $s^*(x)$ is the ratio of the number of buyers to the number of sellers and a seller's equilibrium ex-ante expected net profit is zero.

Proposition 4 is similar to Proposition 2. It is straightforward to prove Proposition 4, so the proof is omitted. Any reserve price in X^* can be supported in equilibrium and given $x \in X^*$ with a unique equilibrium ratio of the number of buyers to the number of sellers. However, the ratio is the lowest when $x = x^*$: Heuristically, the largest number of sellers enter the market when the equilibrium reserve price is equal to the monopoly reserve price. Regardless of the equilibrium reserve price, a seller's equilibrium ex-ante expected net profit is zero.

7 Concluding Remarks

This paper proposes and studies general competing mechanism games of incomplete information, one for markets with frictions and the other for markets without frictions. The approach taken in this paper can be used in various applications such as competing prices, competing auctions and competition in on-line markets. We believe that our approach can also be applied to other problems. For example, Norman (2004) considers public good provision with exclusion for a single mechanism designer. Using the results from this paper we can also consider competing public good provisions with exclusion where buyers eventually select one seller for a public good.

This paper is based on the private value environment in the sense that each buyer's type affects only her payoffs. It is not difficult to imagine an interdependent value environment where a buyer's payoff depends on other buyers' types as well. In this case, one can consider a set of ex-post incentive compatible (EPIC) direct mechanisms (Bergemann and Morris 2005) to punish a deviator. The property of ex-post incentive compatibility also does not depend on the endogenous distribution of the number of participating buyers given that the participating buyers report their true types. Therefore, one can always fix truthful type reporting to non-deviators. Even without the BIC-EPIC equivalence, we can consider robust equilibrium BIC allocations supportable with EPIC punishment: The lower bound of a seller's robust equilibrium ex-ante expected payoff supportable with EPIC punishment is

$$\inf_{\mu \in \Omega_E} \sup_{\mu' \in \mathcal{M}(\mu)} \left(\sup_{\pi(\mu',\mu) \in \Pi(\mu',\mu)} \Phi_J(\mu', \mu, \pi) \right),$$

where Ω_E is the set of all EPIC direct mechanisms and $\mathcal{M}(\mu)$ is the set of all possible BIC direct mechanisms available for a seller conditional on the other sellers' EPIC mechanism $\mu \in \Omega_E$. The upper bound is a seller's payoff associated with a BIC allocation that is derived by a joint profit maximization.

Appendix A. Proof of Corollary 1

Consider any feasible allocation $(\tilde{\mu}, \tilde{\pi}) \in Z$. Therefore, it is a continuation equilibrium that each buyer of type x chooses each seller with equal probability $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) = 1/J$ whenever $\tilde{\pi}(\tilde{\mu}, \tilde{\mu})(x) > 0$ and reports her true type upon selecting a seller. Then, a seller's ex-ante expected payoff and a buyer's interim expected payoff, $\Phi_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi})$ and $U_J(\tilde{\mu}, \tilde{\mu}, \tilde{\pi}, x)$, are derived with the type distribution $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))(x)$.

Let $\tilde{Q}^k$ be the reduced form of $\tilde{q}^k$ in $\tilde{\mu} = \{\tilde{q}^1, \ldots, \tilde{q}^K, \tilde{t}\}$ based on $z(\tilde{\pi}(\tilde{\mu}, \tilde{\mu}))$. Let Q^k be the reduced form of q^k in $\mu = \{q^1, \ldots, q^K, t\}$ based on $z(\pi(\mu, \mu))$ with $\pi(\mu, \mu) = \tilde{\pi}(\tilde{\mu}, \tilde{\mu})$. Then, we define $\tilde{V}(x) := \sum_{k \in \mathcal{K}} b^k \tilde{Q}^k(x)$ and $V(x) := \sum_{k \in \mathcal{K}} b^k Q^k(x)$. If

$\pi(\mu,\mu) = \tilde{\pi}(\tilde{\mu},\tilde{\mu})$, then

$$z(\pi(\mu,\mu)) = z(\tilde{\pi}(\tilde{\mu},\tilde{\mu})) \tag{40}$$

Because $z(\pi(\mu,\mu))$ is a probability distribution over $\bar{X}$, and $\bar{X}$ is the message space for a buyer in a direct mechanism, (40) implies that the probability distribution over $\bar{X}$ for a direct mechanism is preserved and that they are independent of each other. Let $\mathbb{E}[\cdot|z(\pi(\mu,\mu))]$ be the expectation operator over a buyer's type x given the probability distribution $z(\pi(\mu,\mu))$. Given the linear payoff structure, we can apply Theorem 1 and Lemma 3 in Gershkov, et al. (2013) for the case of anonymous (and hence non-discriminatory) mechanisms to show the existence of a DIC direct mechanism μ such that

$$V(x) = \tilde{V}(x) \text{ for all } x, \tag{41}$$

$$\mathbb{E}[Q^k(x)|z(\pi(\mu,\mu))] = \mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))] \text{ for all } k \in \mathcal{K} \tag{42}$$

with the transfers t, which preserves each buyer's interim expected payoff upon selecting the seller. That is,

$$U_J(\mu,\mu,\pi,x) = U_J(\tilde{\mu},\tilde{\mu},\tilde{\pi},x) \text{ for all } x \tag{43}$$

Note that (42) is the "*ex-ante*" probability that alternative k is chosen. Taking the expected value of each side of (43) over x and applying (41) yields

$$\begin{aligned}
&\mathbb{E}[T(x)|z(\pi(\mu,\mu))] \\
&= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))] \\
&+ g^k\left(\sum_{k\in\mathcal{K}}\mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))] - \sum_{k\in\mathcal{K}}\mathbb{E}[Q^k(x)|z(\pi(\mu,\mu))]\right) \\
&= \mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))]
\end{aligned} \tag{44}$$

The first and the second equalities in (44) hold because of (41) and (42) respectively.

On the other hand, the seller's ex-ante expected payoff associated with μ satisfies

$$\begin{aligned}
&\Phi_J(\mu,\mu,\pi) &(45)\\
&= \sum_{k\in\mathcal{K}}(a^k w + y^k)\mathbb{E}[Q^k(x)|z(\pi(\mu,\mu))] - N\times\mathbb{E}[T(x)|z(\pi(\mu,\mu))]\\
&= \sum_{k\in\mathcal{K}}(a^k w + y^k)\mathbb{E}[\tilde{Q}^k(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))] - N\times\mathbb{E}[\tilde{T}(x)|z(\tilde{\pi}(\tilde{\mu},\tilde{\mu}))]\\
&= \Phi_J(\tilde{\mu},\tilde{\mu},\tilde{\pi})
\end{aligned}$$

where the second equality holds because of (42) and (44). Therefore, the seller's ex-ante expected payoff is preserved. For any ϕ in Φ_J^*, we can find a (BIC) allocation $(\tilde{\mu},\tilde{\pi})$ that supports it. (45) implies that it can be supported by a DIC allocation, i.e., $(\mu,\pi)\in Z$ with $\mu\in\Omega_D$.

Because monetary transfer to a participating buyer is restricted to be non-positive, we need to show whether a DIC direct mechanism μ also has the property of "non-positive" monetary transfer to a participating buyer given that the original BIC direct mechanism $\tilde{\mu}$ has that property. Note that given $\tilde{\mu}=\{\tilde{q}^1,\ldots,\tilde{q}^K,\tilde{t}\}$, we have

$$\tilde{T}(\underline{x}) = \int_{\underline{x}}^{\bar{x}}\cdots\int_{\underline{x}}^{\bar{x}}\tilde{t}(\underline{x},s_2,\ldots,s_I)dz(\pi(\tilde{\mu},\tilde{\mu}))(s_2)\ldots dz(\pi(\tilde{\mu},\tilde{\mu}))(s_I)\le 0,$$

because monetary transfers to a participating buyer are non-positive and that

$$\tilde{V}(\underline{x}) = \sum_{k\in\mathcal{K}} b^k\tilde{Q}^k(\underline{x})\ge 0$$

by $b^k\ge 0$ and $\tilde{Q}^k(\underline{x})\ge 0$ for all k. Applying (40) as the underlying probability distribution over the message space in the direct mechanism to Theorem 1 in Gershkov, et al. (2013), monetary transfers in $\mu=\{q^1,\ldots,q^K,t\}$ are determined by

$$t(x,\mathbf{x}_{-1}) = \frac{\rho(\underline{x},\mathbf{x}_{-1})}{\tilde{V}(\underline{x})}\tilde{T}(\underline{x}) + \rho(\underline{x},\mathbf{x}_{-1})\underline{x} - \rho(x,\mathbf{x}_{-1})x + \int_{\underline{x}}^{x}\rho(s,\mathbf{x}_{-1})ds,$$

where ρ is defined as

$$\rho(s,\mathbf{x}_{-1}) := \sum_{k\in\mathcal{K}} b^k q^k(s,\mathbf{x}_{-1})\ge 0$$

for all $s\in X=[\underline{x},\bar{x}]$. Because μ is DIC, $\rho(s,\mathbf{x}_{-1})$ is non-decreasing in s.

First of all, we have

$$t(\underline{x}, \mathbf{x}_{-1}) = \frac{\rho(\underline{x}, \mathbf{x}_{-1})}{\tilde{V}(\underline{x})} \tilde{T}(\underline{x}) \leq 0. \tag{46}$$

because $\tilde{T}(\underline{x}) \leq 0$, $\tilde{V}(\underline{x}) \geq 0$ and $\rho(\underline{x}, \mathbf{x}_{-1}) \geq 0$.[19] Secondly, consider $t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1})$ for any $x, x' \in X$ with $x > x'$:

$$t(x, \mathbf{x}_{-1}) - t(x', \mathbf{x}_{-1}) = -\rho(x, \mathbf{x}_{-1})x + \rho(x', \mathbf{x}_{-1})x' + \int_{x'}^{x} \rho(s, \mathbf{x}_{-1}) ds \leq 0. \tag{47}$$

The inequality holds because we have

$$\rho(x, \mathbf{x})x - \rho(x', \mathbf{x})x' \geq \int_{x'}^{x} \rho(s, \mathbf{x}) ds$$

given that $\rho(s, \mathbf{x}_{-1})$ is non-decreasing in s with $\rho(s, \mathbf{x}_{-1}) \geq 0$ for all $s \in X \subset \mathbb{R}_{+}$. (46) and (47) imply that $t(x, \mathbf{x}_{-1}) \leq 0$ for all $x \in X$ given any $\mathbf{x}_{-1}$.

Finally we can reach the following conclusion. Because $(\tilde{\mu}, \tilde{\pi})$ is an allocation where it is a continuation equilibrium that buyers report their true type to a seller who they select according to $\tilde{\pi}$, (μ, π) is an allocation such that (i) it is a continuation equilibrium that buyers report their true type to a seller who they select according to π and (ii) each buyer's interim expected payoffs and the ex-ante expected payoff for the seller with the original direct mechanism $\tilde{\mu}$ are preserved.

Appendix B: Proof of Theorem 4

First, we prove that $\overline{\phi}_J$ is that which is specified in (29). $\overline{\phi}_J$ is derived by the seller's joint profit maximization as in (10). Since all sellers post an identical direct mechanism $\mu = \{q, p\}$ in the joint profit maximization as their selling mechanism and that each buyer selects a seller according to $\pi(\mu, \mu)(x) = 1/J$, we can then derive a reduced-form direct mechanism $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_{+}$ similar to (4) and (5), based on $z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J}$ for all $x \in [0, 1]$ according to (9). For notational simplicity, let $z(\pi) = z(\pi(\mu, \mu))$ for any μ.

Then, the seller's problem is to find $Q : X \to [0, 1]$ and $P : X \to \mathbb{R}_{+}$ that maximize

[19] As in Gershkov, et al (2013), 0/0 is interpreted as 1.

(31) subject to

$$
\begin{aligned}
\text{(IC) } xQ(x) - P(x) &\geq xQ(x') - P(x') \text{ for all } (x, x') \in [0,1], \\
\text{(IR) } xQ(x) - P(x) &\geq 0 \text{ for all } x \in [0,1].
\end{aligned}
$$

Let the interim expected utility for a buyer with valuation $x_i \in [0,1]$ be denoted by $U(x_i) = x_iQ(x_i) - P(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payment:

$$
\int_0^1 \cdots \int_0^1 x_i q(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \int_0^1 U(x_i) dz(\pi)(x_i) \tag{48}
$$

By using the envelope theorem, we can show that

$$
U(x_i) = U(0) + \int_0^{x_i} Q(s)ds. \tag{49}
$$

Because the seller does not want to leave any unnecessary rents to buyers, we have $U(0) = 0$ at the optimum. Substituting (49) into (48) and integrating by parts yields

$$
\begin{aligned}
& N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \quad (50) \\
= \; & N \int_0^1 \cdots \int_0^1 \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q(x_i, \mathbf{x}_{-i}) dz(\pi)(x_1) \cdots dz(\pi)(x_N),
\end{aligned}
$$

The last equality holds because

$$
\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.
$$

Since the incentive compatibility for buyer i is equivalent to (49), and by the monotonicity of $Q(x_i)$, the optimal mechanism maximizes (50) subject to $q(x_i, \mathbf{x}_{-i}) \geq 0$ and $\sum_{i=1}^N q(x_i, \mathbf{x}_{-i}) \leq 1$ for all $(x_i, \mathbf{x}_{-i}) \in X^N$, and $Q(\cdot)$ is non-decreasing.

Because there is no capacity limit, (50) is maximized if the seller sells its products to all buyers whose valuation is no less than x^*, that is, for all $\mathbf{x}_{-i}$

$$
q(x_i, \mathbf{x}_{-i}) := Q(x_i) = \begin{cases} 1 & \text{if } x_i \geq x^*, \\ 0 & \text{otherwise.} \end{cases} . \tag{51}
$$

Therefore, (50) at the solution for the joint profit maximization is

$$\overline{\phi}_J = \frac{N}{J}\int_{x^*}^{1}\left[\left(x - \frac{1-F(x)}{f(x)}\right)\right] f(x)dx$$

Given (51), a buyer's payment in the optimal selling mechanism becomes (32).

The optimal reduced-form selling mechanism $\mu = \{Q, P\}$, characterized by (51) and (32), is DIC because the probability and payment only depends on the buyer's own type message but not other buyers'. It is also deceptively simple to be implemented: A seller simply posts a single price x^*. Any buyer who wants the product purchases it at price x^*.

To complete the proof, let us show how to support a seller's ex-ante expected profit $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$ in an equilibrium. When each seller sells his product at price x and buyers choose each seller with equal probability, a seller's ex-ante expected profit is

$$\phi = \frac{N}{J}x(1 - F(x)). \tag{52}$$

If $x = 0$ or 1, then $\phi = 0$. If $x = x^*$, then $\phi = \overline{\phi}_J$. By the continuity of (52), then for any $\phi \in \Phi_J^* = [0, \overline{\phi}_J]$, there exists at least one $x(\phi)$ that satisfies (52)

Fix any $\phi \in \Phi_J^*$ as a seller's ex-ante expected profit. Let every seller post a deviation-reporting contract g^* specified in (30) with $H = \{0, 1\}$. If no seller deviates from g^*, all buyers report $h = 0$ to every seller whose price is $x(\phi)$. Then, buyers with valuation $x(\phi)$ or higher chooses some seller with equal probability and buys the product at price $x(\phi)$. This continuation equilibrium behavior yields the ex-ante expected payoff of ϕ for every seller.

Suppose that a seller deviates from g^*. Then, all buyers report $h = 1$ to every non-deviating seller whose price goes down to zero. A deviating seller must offer an (expected) price that is no higher than zero in order to attract any buyers. This is clearly not profitable because $\phi \geq 0$. Therefore, there is no profitable deviation to any arbitrary mechanism.

Appendix C: Proof of Theorem 5

$\overline{\phi}_J$ is reached when sellers jointly maximize their ex-ante expected profits. Let a direct mechanism μ be characterized by $\{q_i, p_i\}_{i=1}^N$, where $q_i : X^N \to [0, 1]$ and $p_i : X^N \to \mathbb{R}_+$ specify the probability of acquiring the object and the payment to the seller, respectively. Specifically, for all $\mathbf{x} = [x_1, \ldots, x_N]$ and all $i = 1, \ldots, N$, $q_i(\mathbf{x})$ and $p_i(\mathbf{x})$ are the probability that buyer i acquires the object and buyer i's payment to the seller with $q_i(0, \mathbf{x}_{-i}) = p_i(0, \mathbf{x}_{-i}) = 0$ for all $\mathbf{x}_{-i} \in X^{N-1}$.

Because every seller offers an identical direct mechanism, each buyer selects each seller with equal probability, $1/J$. Then, the probability that a buyer's valuation is less than x_i or she selects another seller is given by

$$z(\pi(\mu, \mu))(x) = 1 - \frac{1}{J} + \frac{F(x)}{J} \text{ for all } x. \tag{53}$$

Because each seller offers identical direct mechanisms, we can fix $z(\pi(\mu, \mu))(x_i)$ to (53) in the seller's joint profit maximization problem. For simplicity, we define

$$z(\pi)(x_i) = z(\pi(\mu, \mu))(x_i)$$

for all μ and all x_i. Then, we can derive the reduced-form direct mechanism $\{Q_i, P_i\}_{i=1}^N$ similar to (4) and (5) based on $z(\pi(\mu, \mu))$ specified in (53). It follows that the seller's joint profit maximization problem is to find a direct mechanism that maximizes her ex-ante expected profit:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N p_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \tag{54}$$

subject to

$$\begin{aligned} \text{(IC) } x_i Q_i(x_i) - P_i(x_i) &\geq x_i Q_i(x_i') - P_i(x_i') \text{ for all } (x_i, x_i') \in [0, 1], \\ \text{(IR) } x_i Q_i(x_i) - P_i(x_i) &\geq 0 \text{ for all } x_i \in [0, 1], \end{aligned}$$

and $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$.

Let buyer i's interim expected utility be denoted by $U_i(x_i) = x_i Q_i(x_i) - P_i(x_i)$. The seller's objective function can be rewritten as a function of the buyers' interim expected utilities by substituting for the payments:

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N x_i q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) - \sum_{i=1}^N \int_0^1 U_i(x_i) dz(\pi)(x_i) \tag{55}$$

In (49, we have $U_i(0) = 0$ at the optimum because the seller does not want to leave any unnecessary rents to buyers. Substituting (49) with $U_i(0) = 0$ into (55) and integrating by parts yields

$$\int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N) \tag{56}$$

$$= \int_0^1 \cdots \int_0^1 \sum_{i=1}^N \left(x_i - \frac{1 - F(x_i)}{f(x_i)} \right) q_i(\mathbf{x}) dz(\pi)(x_1) \cdots dz(\pi)(x_N), \tag{57}$$

The last equality holds because

$$\frac{1 - z(\pi)(x_i)}{z'(\pi)(x_i)} = \frac{\frac{1}{J} - \frac{F(x_i)}{J}}{\frac{f(x_i)}{J}} = \frac{1 - F(x_i)}{f(x_i)}.$$

Because the incentive compatibility for buyer i is equivalent to (49), and by the monotonicity of $Q_i(x_i)$, the optimal mechanism maximizes (57) subject to $q_i(\mathbf{x}) \geq 0$ and $\sum_{i=1}^N q_i(\mathbf{x}) \leq 1$ for all $\mathbf{x}$, and $Q_i(\cdot)$ is non-decreasing.

Because a mechanism is anonymous ($Q_i(\cdot) = Q(\cdot)$), the optimal mechanism takes the form of an auction with reserve price: If the virtual valuation, $x_i - \frac{1-F(x_i)}{f(x_i)}$, is non-decreasing, the object goes to the highest valuation buyer if it is sold at all. It is sold if and only if

$$\max_{i \in \{1,\ldots,N\}} x_i \geq x^*, \text{ with } x^* \text{ defined in (33).}$$

Because the auction is anonymous, the seller's maximum ex-ante expected profit is

$$\begin{aligned} \overline{\phi}_J &= N \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) Q(x) z'(\pi)(x) dx \\ &= \frac{N}{J} \int_{x^*}^1 \left(x - \frac{1 - F(x)}{f(x)} \right) \left(1 - \frac{1 - F(x)}{J} \right)^{N-1} f(x) dx. \end{aligned}$$

As we take its limit, $\overline{\phi}_\infty = \lim_{J \to \infty} \overline{\phi}_J$ is equal to the expression in (35).

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