Questions and Answers on Asymptotic Theory

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1. Let x_i be scalar random variables with $\mathrm{E}(x_i) = \mu$ and $\mathrm{Var}(x_i) = \sigma^2$. x_i is observed for $i = 1, \ldots, n$. They are independently distributed. For each of the following functions,

(i) state whether it converges in probability, or converges in distribution, or does neither.

(ii) give its probability limit if it converges in probability, or give its limiting distribution (as $n \to \infty$) if it converges in distribution.

(a) $n^{-1} \sum_{i=1}^{n} x_i$

(b) $n^{-1} \sum_{i=1}^{n} x_i^2$

(c) $n^{1/2}(\bar{x} - \mu)$, where $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$.

(d) $\bar{x}^2(n^{1/2}(\bar{x} - \mu))$

2. Let x_i and y_i be scalar random variables with the properties $E(x_i) = \mu_x$, $\mathrm{Var}(x_i) = \sigma_x^2$, $E(y_i) = \mu_y$, and $\mathrm{Var}(y_i) = \sigma_y^2$. x_i and y_i are observed for $i = 1, \ldots, n$. They are independently distributed. Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$ and $\bar{y} = n^{-1} \sum_{i=1}^{n} y_i$. For each of the following functions, give its probability limit if it converges in probability, or give its limiting distribution (as $n \to \infty$) if it converges in distribution.

(a) $\bar{x}$

(b) $n^{1/2}(\bar{y} - \mu_y)$

(c) $\bar{x} \times n^{1/2}(\bar{y} - \mu_y)$

(d) $n(\bar{y} - \mu_y)^2/\sigma_y^2$

3. Let x_i be a set of K-element random vectors drawn from a distribution having the property $\mathrm{E}(x_i x_i') = Q$, where Q is a positive definite $K \times K$ matrix. Assume the draws are statistically independent, $i = 1, ..., n$.
Let $y_i = x_i'\beta + \epsilon_i$.
Let ϵ be the $n \times 1$ vector of ϵ_i's. Assume $\mathrm{E}(\epsilon|X) = 0$ and $\mathrm{E}(\epsilon\epsilon'|X) = \sigma^2 I$, but ϵ_i is not necessarily normally distributed.
Let y be the $n \times 1$ vector of y_i's and let X be an $n \times K$ matrix with i^{th} row equal to x_i'.

For each of the following,

(i) state whether it converges in probability, or converges in distribution, or does neither.

(ii) give its probability limit if it converges in probability, or give its limiting distribution (as $n \to \infty$) if it converges in distribution.

No explanation is required.

(a) b, where $b = (X'X)^{-1}X'y$

(b) $3b$

(c) $n^{1/2}(b - \beta)$

(d) s^2, where $s^2 = e'e/(n - K)$ and $e = y - Xb$

(e) $\hat{\sigma}^2$, where $\hat{\sigma}^2 = e'e/n$

(f) $n^{-1/2}(X'X)(b - \beta)$

4. Let x_i be a scalar random variable with $\mathrm{E}(x_i) = \mu$ and $\mathrm{Var}(x_i) = \sigma^2$. x_i is observed for $i = 1, \ldots, n$. The x_i's are independently distributed. Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$.

(a) Using these definitions, state

(i) the Law of Large Numbers (LLN)

(ii) the Central Limit Theorem (CLT)

(b) Let a_n be a random variable which is a function of a data set with sample size n. State what, if anything, is implied about the limits of $E(a_n)$ and $\mathrm{Var}(a_n)$ as $n \to \infty$ by each of the following statements, considered one at a time.

(i) $\mathrm{plim}\ a_n = \mu$.

(ii) $a_n \xrightarrow{d} N[0, \sigma^2]$.

5. Let x_i be a scalar random variable with $\mathrm{E}(x_i) = \mu$ and $\mathrm{Var}(x_i) = \sigma^2$. x_i is observed for $i = 1, \ldots, n$. The x_i's are independently distributed. Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i$.

For each of the following functions, determine its plim if it converges in probability, or state what you can about the mean and variance of its limiting distribution if it converges in distribution.

(a) $\bar{x}^{1/2}$, assuming that $\mu > 0$.

(b) $\bar{x} \times (n^{1/2}(\bar{x} - \mu))$

6. Let $x_i = \begin{bmatrix} x_{1i} \\ x_{2i} \\ x_{3i} \end{bmatrix}$ be a random vector with $E(x_i) = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\text{Var}(x_i) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

x_i is observed for $i = 1, \ldots, n$. The x_i's are independently distributed.

Let $\bar{x} = n^{-1} \sum_{i=1}^{n} x_i = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{bmatrix}$.

Using the multivariate central limit theorem and properties of means and variances of linear combinations of random variables, obtain an asymptotic approximation to the distribution of $\bar{x}_1 + 2\bar{x}_3$ when $n = 100$.

Answers

1. (a) (i) $n^{-1} \sum_{i=1}^{n} x_i$ converges in probability

(ii) $\text{plim}(n^{-1} \sum_{i=1}^{n} x_i) = \text{E}(x_i) = \mu$

(b) (i) $n^{-1} \sum_{i=1}^{n} x_i^2$ converges in probability.

(ii) $\text{plim}(n^{-1} \sum_{i=1}^{n} x_i^2) = \text{E}(x_i^2) = \mu^2 + \sigma^2$.

(c) (i) $n^{1/2}(\bar{x} - \mu)$ converges in distribution

(ii) $\text{N}[0, \text{Var}(x_i)]$.

(d) This function is the one in part (c) multiplied by $\bar{x}^2$. From (a) we know that $\text{plim}(\bar{x}) = \mu$. So as $n \to \infty$, the answer is the same as the answer to $\mu^2(n^{1/2}(\bar{x} - \mu))$. This is just a constant, μ^2, times the function in (c). The answer will be the same as the answer for (c), except the variance of that asymptotic distribution is multiplied by the square of this constant.

(i) converges in distribution

(ii) $\text{N}[0, \mu^4 \text{Var}(x_i)]$.

2. (a) $\text{plim}(\bar{x}) = \mu_x$

(b) $n^{1/2}(\bar{y} - \mu_y) \longrightarrow N[0, \sigma_y^2]$ as $n \to \infty$

(c) $\bar{x} \times n^{1/2}(\bar{y} - \mu_y) \longrightarrow \mu_x \times N[0, \sigma_y^2]$ as $n \to \infty$, and this is $N[0, \mu_x^2 \sigma_y^2]$

(d) $n(\bar{y} - \mu_y)^2/\sigma_y^2 = (\sqrt{n}(\bar{y} - \mu_y)/\sigma_y)^2$. From (b), we know that $\sqrt{n}(\bar{y} - \mu_y) \longrightarrow N[0, \sigma_y^2]$ as $n \to \infty$. Therefore $\sqrt{n}(\bar{y} - \mu_y)/\sigma_y \longrightarrow N[0, 1]$ as $n \to \infty$. We want the limiting distribution of the square of this quantity, which is the distribution of the square of a standard normal random variable, which is a χ^2 with 1 d.f.

3. (a) (i) converges in probability

(ii) β

(b) (i) converges in probability

(ii) 3β

(c) (i) converges in distribution

(ii) $N[0, \sigma^2 Q^{-1}]$

(d) (i) converges in probability

(ii) σ^2

(e) (i) converges in probability

(ii) σ^2

(f) (i) converges in distribution

(ii) write it as $(n^{-1}(X'X))n^{1/2}(b-\beta)$. Since $\text{plim}(n^{-1}X'X) = Q$ and $n^{1/2}(b-\beta) \rightarrow N[0, \sigma^2 Q^{-1}]$, then $n^{-1/2}(X'X)(b-\beta) \rightarrow Q \times$ (a $N[0, \sigma^2 Q^{-1}]$ vector) which is distributed as $N[0, Q(\sigma^2 Q^{-1})Q']$ or $N[0, \sigma^2 Q]$. (Since $X'X$ is symmetric, so is Q.)

4. (a) (i) LLN: $\text{plim } \bar{x} = \mu$

(ii) CLT: $n^{1/2}(\bar{x} - \mu) \xrightarrow{d} N[0, \sigma^2]$

(b) (i) $\lim E(a_n) = \mu$

$\lim \text{Var}(a_n) = 0$

(ii) $\lim E(a_n) = 0$

$\lim \text{Var}(a_n) = \sigma^2$

5. (a) $\text{plim } \bar{x} = \mu$, therefore $\text{plim } \bar{x}^{1/2} = (\text{plim } \bar{x})^{1/2} = \mu^{1/2}$ (Slutsky's theorem).

(b) $\text{plim } \bar{x} = \mu$ and $n^{1/2}(\bar{x} - \mu) \xrightarrow{d} N[0, \sigma^2]$.
Therefore $\bar{x} \times (n^{1/2}(\bar{x} - \mu)) \xrightarrow{d} \mu \times N[0, \sigma^2]$, or $\xrightarrow{d} N[0, \mu^2\sigma^2]$, or $\xrightarrow{d} \mu\sigma z$ where z is a standard normal random variable.

6. Multivariate CLT: $n^{1/2}(\bar{x}-\mu) \xrightarrow{d} N[0, \text{Var}(x_i)]$, where $\mu = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ and $\text{Var}(x_i) = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}$.

Then $\bar{x} \overset{approx}{\longrightarrow} N[\mu, n^{-1}\text{Var}(x_i)] \overset{approx}{\longrightarrow} N\left[\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, n^{-1}\begin{bmatrix} 2 & 0 & 1 \\ 0 & 5 & 1 \\ 1 & 1 & 3 \end{bmatrix}\right]$.

When $n = 100$, then $\bar{x} \overset{approx}{\longrightarrow} N\left[\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} .02 & 0 & .01 \\ 0 & .05 & .01 \\ .01 & .01 & .03 \end{bmatrix}\right]$.

Therefore for any nonrandom vector q: $q'\bar{x} \overset{approx}{\longrightarrow} N\left[\left(q'\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}\right), \left(q'\begin{bmatrix} .02 & 0 & .01 \\ 0 & .05 & .01 \\ .01 & .01 & .03 \end{bmatrix} q\right)\right]$.

Here, $q = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, so that $q'\mu = 3+4 = 7$, and $q'\begin{bmatrix} .02 & 0 & .01 \\ 0 & .05 & .01 \\ .01 & .01 & .03 \end{bmatrix} q = .02+4(.01)+4(.03) = .18$.

Therefore

$\bar{x}_1 + 2\bar{x}_3 \overset{approx}{\longrightarrow} N\,[7, 0.18]$