Questions and Answers on Regression Models with Lagged Dependent Variables and ARMA models

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1. Consider an AR(1) process: $\epsilon_t = \rho\epsilon_{t-1} + u_t$, where $E(u_t) = 0$, $E(u_t^2) = \sigma_u^2$, and $E(u_t u_s) = 0$ for all $t \neq s$. Assume that ϵ_t is stationary. Derive a formula for $\text{Cov}(\epsilon_t, \epsilon_{t-s})$, the covariance of ϵ_t and ϵ_{t-s}, that holds for $s = 0, 1, 2, 3, \ldots$.

2. Let u_t be white noise. That is,

$$\begin{aligned} E(u_t) &= 0 \quad \text{for all } t \\ E(u_t^2) &= \sigma^2 \quad \text{for all } t \\ E(u_t u_{t-s}) &= 0 \quad \text{for all } t \text{ and } s \text{ where } s \neq 0 \end{aligned}$$

For each of the following time series processes, determine the variance of y_t as a function of σ_u^2 and of parameters appearing in the equations below. Also derive the first- and second-order autocovariances and autocorrelations. Assume that the time series processes are stationary.

(a) $y_t = \beta y_{t-1} + u_t$ (y_t is an AR(1) process)

(b) $y_t = \beta + \epsilon_t$, where $\epsilon_t = \rho\epsilon_{t-1} + u_t$ (ϵ_t is an AR(1) process)

(c) $y_t = u_t + \theta u_{t-1}$ (y_t is an MA(1) process)

(d) $y_t = u_t + 0.6u_{t-1} + 0.2u_{t-2} + 0.1u_{t-3}$ (y_t is an MA(3) process)

3. Consider a stationary AR(2) process

$$y_t = \mu + \rho_1 y_{t-1} + \rho_2 y_{t-2} + u_t$$

where $\rho_2 \neq 0$. Are there values of ρ_1 and ρ_2 for which this process could be re-written in moving average form as an MA(2) process? If so, what are the values of ρ_1 and ρ_2? If no such values exist, briefly explain why not.

4. An autoregressive distributed lag model is estimated as:

$$y_t = 31.2 + 0.61y_{t-1} + 0.19y_{t-2} + 1.40x_t + 0.58x_{t-1} + u_t$$

Consider the effect on y of a one-unit increase in x at time t^* in the following two cases:

(a) x remains one unit higher permanently after time t^*.

(b) x immediately returns to its former level at time $t^* + 1$.

Obtain the estimated effect on y in each of these cases at the four time periods: t^*, $t^* + 1$, $t^* + 2$, and the long run effect, $t^* + \infty$.

5. Consider a regression model with a constant term and three explanatory variables, which include the lagged dependent variable y_{t-1} and two other variables, x_{1t} and x_{2t}. The estimated model is

$$y_t = 21.0 + 0.6y_{t-1} + 1.5x_{1t} + 0.75x_{2t} + e_t$$

(a) Obtain the estimated effect on y of a permanent one-unit increase in x_1 at time t^* (that is, x_1 remains one unit higher permanently after time t^*) at the four time periods: t^*; $t^* + 1$; $t^* + 2$; and the long run effect, $t^* + \infty$.

(b) Compare the size of the estimated effect on y of a permanent one-unit increase in x_1 to the size of the estimated effect on y of a permanent one-unit increase in x_2. Mention their initial (time t^*) effects and their long run effects. No algebra or calculations are required.

6. For each of the following time series processes

(a) $y_t = \mu + \beta y_{t-1} + u_t$

(b) $y_t = \mu + u_t + 0.6u_{t-1} + 0.2u_{t-2}$

derive

(i) the unconditional mean, $E(y_t)$

(ii) the unconditional variance, $\text{Var}(y_t)$

(iii) the first-order autocovariance, $\text{Cov}(y_t, y_{t-1}) = E(y_t - E(y_t))(y_{t-1} - E(y_{t-1}))$

Assume: $E(u_t) = 0$ for all t; $E(u_t^2) = \sigma^2$ for all t; $E(u_t u_{t-s}) = 0$ for all t and s where $s \neq 0$; and that the time series processes are stationary.

7. An autoregressive distributed lag model is estimated as

$$y_t = 11 + 0.7y_{t-1} - 0.4y_{t-2} + 9x_t + 2x_{t-1} + u_t$$

Consider the effect on y of a one-unit increase in x at time t^* where x remains one unit higher permanently after time t^*. Obtain the estimated effect on y at time t^*, $t^* + 1$, $t^* + 2$, and the long run effect.

8. Consider a regression model with a constant term and three explanatory variables, which include the lagged dependent variable y_{t-1} and two other variables, x_{1t} and x_{2t}. The estimated model is

$$y_t = 2.1 + 0.8y_{t-1} - 2.0x_{1t} + 0.5x_{2t} + e_t$$

(a) Obtain the estimated effect on y of a permanent one-unit increase in x_1 at time t^* (that is, x_1 remains one unit higher permanently after time t^*) at the four time periods: t^*; $t^* + 1$; $t^* + 2$; and the long run effect, $t^* + \infty$.

(b) Compare the size of the estimated effect on y of a permanent one-unit increase in x_1 with the size of the estimated effect on y of a permanent one-unit increase in x_2. Mention their initial (time t^*) effects and their long run effects. No algebra or calculations are required.

9. Suppose ϵ_t follows a stationary AR(1) process:

$$\epsilon_t = \rho\epsilon_{t-1} + u_t, \qquad t = 1, \ldots, n$$

where u_t is white noise. Let $\rho = 0.6$ and $\text{Var}(u_t) = 5$.

(a) What is the numerical value of the correlation between ϵ_t and ϵ_{t-3}

(b) What is the numerical value of $\text{Var}(\epsilon_t)$

(c) Suppose that $E(u_t) = 10$, instead of the usual zero-mean assumption. What is the numerical value of $E(\epsilon_t)$?

10. Let u_t be white noise, where

$$\begin{aligned} E(u_t) &= 0 \quad \text{for all } t \\ E(u_t^2) &= 20 \quad \text{for all } t \\ E(u_t u_{t-s}) &= 0 \quad \text{for all } t \text{ and } s \text{ where } s \neq 0 \end{aligned}$$

Let $y_t = u_t + 0.7u_{t-1} + 0.1u_{t-2}$. Determine the numerical values of

(a) $\text{Var}(y_t)$

(b) The correlation between y_t and y_{t-1}

(c) The covariance between y_t and y_{t-1}

Answers

1.

$$(1 - \rho L)\epsilon_t = u_t \quad \Rightarrow \quad \epsilon_t = (1 - \rho L)^{-1}u_t$$

$$\epsilon_t = u_t + \rho u_{t-1} + \rho^2 u_{t-2} + \ldots$$

Since $E(\epsilon_t) = 0$ for all t, then

$$\begin{aligned}\text{Cov}(\epsilon_t, \epsilon_{t-s}) &= E(\epsilon_t \epsilon_{t-s}) \\ &= E(u_t + \rho u_{t-1} + \rho^2 u_{t-2} + \ldots + \rho^s u_{t-s} + \ldots) \times (u_{t-s} + \rho u_{t-s-1} + \rho^2 u_{t-s-2} + \ldots)\end{aligned}$$

Because $(E u_t u_s) = 0$ for all $t \neq s$, the only terms with non-zero expectations in this product are those with equal subscripts on the u's. Then the above expression simplifies to

$$\begin{aligned}\text{Cov}(\epsilon_t, \epsilon_{t-s}) &= E(\rho^s u_{t-s}^2 + \rho^{s+2} u_{t-s-1}^2 + \rho^{s+4} u_{t-s-2}^2 + \ldots) \\ &= \rho^s(\sigma_u^2 + \rho^2 \sigma_u^2 + \rho^4 \sigma_u^2 + \ldots) \\ &= \rho^s \sigma_u^2 (1 + \rho^2 + \rho^4 \ + \ldots) \\ &= \frac{\rho^s}{(1-\rho^2)} \sigma_u^2 \quad , \quad \text{for all } s = 0, 1, 2, \ldots\end{aligned}$$

2. (a) $\text{Var}(y_t) = \text{Var}(\beta y_{t-1} + u_t)$

$$\begin{aligned}&= \beta^2 \text{Var}(y_{t-1}) + \sigma_u^2 \\ &= \beta^2 \text{Var}(y_t) + \sigma_u^2\end{aligned}$$

so $\text{Var}(y_t) = \sigma_u^2/(1-\beta^2)$

$$\begin{aligned}\text{Cov}(y_t, y_{t-1}) &= E(y_t \times y_{t-1}) \text{ (since } E y_t = 0) \\ &= E(\beta y_{t-1} + u_t) y_{t-1} = \beta E(y_{t-1}^2) = \beta \text{Var}(y_t)\end{aligned}$$

For $\text{Cov}(y_t, y_{t-2})$, use: $y_t = \beta y_{t-1} + u_t = \beta(\beta y_{t-2} + u_{t-1}) + u_t = \beta^2 y_{t-2} + \beta u_{t-1} + u_t$
Then

$$\begin{aligned}\text{Cov}(y_t, y_{t-2}) &= E(y_t y_{t-2}) = E(\beta^2 y_{t-2} + \beta u_{t-1} + u_t) y_{t-2} \\ &= \beta^2 E(y_{t-2}^2) = \beta^2 \text{Var}(y_t)\end{aligned}$$

Substitutions then give:

$$\text{Corr}(y_t, y_{t-1}) = \frac{\text{Cov}(y_t, y_{t-1})}{\sqrt{\text{Var}(y_t)\text{Var}(y_{t-1})}} = \beta$$

and similarly

$$\text{Corr}(y_t, y_{t-2}) = \beta^2$$

(b) $(y_t - \beta) = \epsilon_t = \rho\epsilon_{t-1} + u_t$

$$= \rho(y_{t-1} - \beta) + u_t$$

This is like (a) except now $E(y_t) = \beta$ instead of $= 0$ and we now have ρ replacing part (a)'s β. Then

$$\begin{aligned}
\text{Var}(y_t) &= \sigma_u^2/(1-\rho^2) \\
\text{Cov}(y_t, y_{t-1}) &= E(y_t - \beta)(y_{t-1} - \beta) = \rho \text{Var}(y_t) \\
\text{Cov}(y_t, y_{t-2}) &= \rho^2 \text{Var}(y_t) \\
\text{Corr}(y_t, y_{t-1}) &= \rho \\
\text{Corr}(y_t, y_{t-2}) &= \rho^2
\end{aligned}$$

(c) $\text{Var}(y_t) = \text{Var}(u_t + \theta u_{t-1}) = \text{Var}(u_t) + \theta^2 \text{Var}(u_{t-1}) = \sigma_u^2 + \theta^2\sigma_u^2 = (1+\theta^2)\sigma_u^2$

$$\text{Cov}(y_t, y_{t-1}) = E(u_t + \theta u_{t-1})(u_{t-1} + \theta u_{t-2}) = \theta E(u_{t-1}^2) = \theta\sigma_u^2$$

$\text{Cov}(y_t, y_{t-2}) = 0$ (y_t and y_{t-2} have no u_t's in common and the u_t's are not correlated)

$$\text{Corr}(y_t, y_{t-1}) = \frac{\theta\sigma_u^2}{(1+\theta^2)\sigma_u^2} = \frac{\theta}{1+\theta^2}$$

$$\text{Corr}(y_t, y_{t-2}) = 0$$

(d) $\text{Var}(y_t) = \sigma_u^2 + (0.6)^2\sigma_u^2 + (0.2)^2\sigma_u^2 + (0.1)^2\sigma_u^2$

$$\begin{aligned}
&= (1 + 0.36 + 0.04 + 0.01)\sigma_u^2 \\
&= 1.41\sigma_u^2
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(y_t, y_{t-1}) &= E(u_t + 0.6u_{t-1} + 0.2u_{t-2} + 0.1u_{t-3})(u_{t-1} + 0.6u_{t-2} + 0.2u_{t-3} + 0.1u_{t-4}) \\
&= 0.6\sigma_u^2 + 0.12\sigma_u^2 + 0.02\sigma_u^2 \\
&= 0.74\sigma_u^2
\end{aligned}$$

$$\begin{aligned}
\text{Cov}(y_t, y_{t-2}) &= E(u_t + 0.6u_{t-1} + 0.2u_{t-2} + 0.1u_{t-3})(u_{t-2} + 0.6u_{t-3} + 0.2u_{t-4} + 0.1u_{t-5}) \\
&= 0.2\sigma_u^2 + 0.06\sigma_u^2 \\
&= 0.26\sigma_u^2
\end{aligned}$$

$$\text{Corr}(y_t, y_{t-1}) = \frac{0.74}{1.41} = 0.52$$

$$\text{Corr}(y_t, y_{t-2}) = \frac{0.26}{1.41} = 0.18$$

3. Write this process as

$$(1 - \rho_1 L - \rho_2 L^2) y_t = \mu + u_t$$

Invert the lag polynomial to get it in MA form

$$y_t = (1 - \rho_1 L - \rho_2 L^2)^{-1}(\mu + u_t) = \mu/(1 - \rho_1 - \rho_2) + (1 - \rho_1 L - \rho_2 L^2)^{-1} u_t$$

The inverse lag polynomial $(1 - \rho_1 L - \rho_2 L^2)^{-1}$ is an infinite series of the form $1 + \theta_1 L + \theta_2 L^2 + \theta_3 L^3 + \ldots$, which is an infinite-order MA, not an MA(2). One way to see this is to factor the original quadratic lag polynomial as $(1 - \lambda_1 L)(1 - \lambda_2 L)$ for some λ_1 and λ_2 values. λ_1 and λ_2 both are non-zero since $\rho_2 \neq 0$. The inverse of this factorized lag polynomial is the product of two infinite-term geometric series

$$(1 - \lambda_1 L)^{-1}(1 - \lambda_2 L)^{-1} = (1 + \lambda_1 L + \lambda_1^2 L^2 + \lambda_1^3 L^3 + \ldots)(1 + \lambda_2 L + \lambda_2^2 L^2 + \lambda_2^3 L^3 + \ldots)$$

which itself is an infinite series.

4. (Note that Δ represents the change in y due to a change in x. It does not represent the first-difference operator here.)

(a) $\Delta y_{t^*} = 1.40 \Delta x_{t^*} = 1.40(1) = 1.40$

$$\begin{aligned} \Delta y_{t^*+1} &= 0.61 \Delta y_{t^*} + 1.40 \Delta x_{t^*+1} + 0.58 \Delta x_{t^*} \\ &= 0.61(1.40) + 1.40(1) + 0.58(1) \\ &= 2.834 \end{aligned}$$

$$\begin{aligned} \Delta y_{t^*+2} &= 0.61 \Delta y_{t^*+1} + 0.19 \Delta y_{t^*} + 1.40 \Delta x_{t^*+2} + 0.58 \Delta x_{t^*+1} \\ &= 0.61(2.834) + 0.19(1.40) + 1.40(1) + 0.58(1) \\ &= 3.975 \end{aligned}$$

The permanent effect can be obtained from $\Delta y = 0.61 \Delta y + 0.19 \Delta y + 1.40 \Delta x + 0.58 \Delta x$, where Δx is the permanent change in x. Then solve for Δy:

$$(1 - 0.61 - 0.19) \Delta y = 1.98 \Delta x$$

$$\Delta y = \frac{1.98}{0.2} \Delta x = 9.9 \Delta x = 9.90$$

(b) $\Delta y_{t^*} = 1.40 \Delta x_{t^*} = 1.40$

$$\begin{aligned}\Delta y_{t^*+1} &= 0.61 \Delta y_{t^*} + 0.58 \Delta x_{t^*} \quad (\text{Now } \Delta x_{t^*+1} = 0) \\ &= 0.61(1.40) + 0.58(1) \\ &= 1.434\end{aligned}$$

$$\begin{aligned}\Delta y_{t^*+2} &= 0.61 \Delta y_{t^*+1} + 0.19 \Delta y_{t^*} \\ &= 0.61(1.434) + 0.19(1.40) \\ &= 1.141\end{aligned}$$

The permanent effect is $\Delta y = 0$ since the permanent change in x is $\Delta x = 0$.

5. (a) Effect at time $t* : 1.5$
at time $t * +1 : 1.5 + 0.6 \times 1.5 = 2.4$
at time $t * +2 : 2.4 + (0.6)^2 \times 1.5 = 2.94$
at time $t * +\infty : 1.5/(1 - .6) = 3.75$

(b) At every time period, the effects of x_2 on y are half as big as the effects of x_1 on y. Reason: The coefficient on x_2 is half the size of the coefficient on x_1, and the dynamic pattern of the effects is the same for both, because that depends only on the coefficient on y_{t-1}.

6. (a) (i) $Ey_t = \mu + \beta Ey_{t-1} + Eu_t \implies Ey_t = \mu + \beta Ey_t \implies (Ey_t)(1 - \beta) = \mu$
$\implies Ey_t = \mu/(1 - \beta)$

(ii) $\text{Var}(y_t) = \beta^2 \text{Var}(y_{t-1}) + \text{Var}(u_t) \implies \text{Var}(y_t) = \beta^2 \text{Var}(y) + \sigma^2$
$\implies \text{Var}(y) = \sigma^2/(1 - \beta^2)$

(iii) $y_t - Ey_t = \mu + \beta y_{t-1} + u_t - E(\mu + \beta y_{t-1} + u_t) = \mu + \beta y_{t-1} + u_t - (\mu + \beta Ey_{t-1})$
$= \beta(y_{t-1} - Ey_{t-1}) + u_t$
So $E(y_t - Ey_t)(y_{t-1} - E(y_{t-1})) = E(\beta(y_{t-1} - Ey_{t-1}) + u_t)(y_{t-1} - Ey_{t-1})$
$= \beta E(y_{t-1} - Ey_{t-1})^2 = \beta \text{Var}(y_t)$

(b) (i) $Ey_t = \mu$

(ii) $\text{Var}(y) = E(y_t - \mu)^2 = (1 + .6^2 + .2^2)\sigma^2 = 1.4\sigma^2$

(iii) $E(y_t - Ey_t)(y_{t-1} - E(y_{t-1})) = E(u_t + .6u_{t-1} + .2u_{t-2})(u_{t-1} + .6u_{t-2} + .2u_{t-3})$
$= .6Eu_{t-1}^2 + .12Eu_{t-2}^2 = .72\sigma^2$

7. at t^* $\Delta y = 9 \times 1 = 9$
at $t^* + 1$, $\Delta y = 0.7 \times 9 + 9 \times 1 + 2 \times 1 = 17.3$
at $t^* + 2$, $\Delta y = 0.7 \times 17.3 - 0.4 \times 9 + 9 \times 1 + 2 \times 1 = 19.51$
long run effect is $\Delta y = \frac{9+2}{1-.7+.4} = \frac{11}{0.7} = 15.71$

8. (a) Effect at time $t*: -2.0$

at time $t*+1: -2.0 + 0.8 \times (-2.0) = -3.6$

at time $t*+2: -2.0 + 0.8 \times (-3.6) = -4.88$

at time $t*+\infty: -2.0/(1-.8) = -10.0$

(b) At every time period, the effect of a change in x_2 on y is -0.25 times the effect of a change in x_1 on y. This is because the coefficient on x_2 is -0.25 times the coefficient on x_1. This ratio does not change over time, because the way that the effect changes over time in this model depends only on the coefficient on y_{t-1}, in the same way for both the x_1 and x_2 effects.

9. (a) When ϵ_t follows a stationary AR(1) process with first-order autocorrelation coefficient ρ, then $\text{Corr}(\epsilon_t, \epsilon_{t-s}) = \rho^s$. Therefore $\text{Corr}(\epsilon_t, \epsilon_{t-3}) = \rho^3 = (0.6)^3 = 0.216$

(b) $\text{Var}(\epsilon_t) = \rho^2 \text{Var}(\epsilon_{t-1}) + \text{Var}(u_t)$

$\text{Var}(\epsilon_t) = 0.36\text{Var}(\epsilon_t) + 5$

$\text{Var}(\epsilon_t) = \frac{5}{1-0.36} = 7.81$

(c) $E(\epsilon_t) = \rho E(\epsilon_{t-1}) + E(u_t)$

$E(\epsilon_t) = 0.6E(\epsilon_t) + 10$

$E(\epsilon_t) = \frac{10}{1-0.6} = 25$

10. (a)

$$\begin{aligned} \text{Var}(y_t) &= \text{Var}(u_t) + (.7)^2\text{Var}(u_{t-1}) + (.1)^2\text{Var}(u_{t-2}) \\ &= 20 + .49(20) + (.01)20 \\ &= 20(1+.5) = 30 \end{aligned}$$

(b) Since $E(y_t) = 0$, then

$$\begin{aligned} \text{Cov}(y_t, y_{t-1}) &= E(y_t y_{t-1}) \\ &= E(u_t + .7u_{t-1} + .1u_{t-2})(u_{t-1} + .7u_{t-2} + .1u_{t-3}) \\ &= .7E(u_{t-1}^2) + (.7)(.1)E(u_{t-2}^2) \\ &= (.7 + .07)20 = 15.4 \end{aligned}$$

(c)

$$\begin{aligned} \text{Corr}(y_t, y_{t-1}) &= \frac{\text{Cov}(y_t, y_{t-1})}{\sqrt{\text{Var}(y_t)\text{Var}(y_{t-1})}} \\ &= \frac{15.4}{\sqrt{30 \times 30}} = .513 \end{aligned}$$