Unit Roots and Characteristic Roots

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One way to determine if an AR(p) process such as:

$$y_t = \rho_1 y_{t-1} + \rho_2 y_{t-2} + \ldots + \rho_p y_{t-p} + \epsilon_t$$

is stationary is to look at the roots of the *characteristic equation*. This equation is obtained by expressing the process in lag polynomial notation:

$$(1 - \rho_1 L - \rho_2 L^2 - \ldots - \rho_p L^p) y_t = \epsilon_t$$

To get the characteristic equation, replace the lag operator L by a variable (call it z), and set the resulting polynomial equal to zero:

$$1 - \rho_1 z - \rho_2 z^2 - \ldots - \rho_p z^p = 0$$

The *characteristic roots* are the values of z that solve this equation. There are p of them, although some of them may be equal. y_t is *stationary* if all of the roots "lie outside the unit circle". This phrase reflects the fact that some of these roots may be complex numbers. If the roots all are real numbers (that is, none of the roots are complex numbers), then we can say that y_t is stationary if the absolute values of all of these real roots are greater than one. If a root equals one or minus one, it is called a *unit root*. If there is at least one unit root, or if any root lies between plus and minus one, then the series is not stationary.

For example, the AR(1) process: $y_t = \rho_1 y_{t-1} + \epsilon_t$ has a characteristic equation: $1 - \rho_1 z = 0$ and its one characteristic root is $z^* = 1/\rho_1$. The series is stationary as long as $|\rho_1| < 1$ which is the same condition as $|z^*| > 1$.

For $ARMA(p, q)$ processes, the $MA(q)$ part is irrelevant for determining stationarity, so the $MA(q)$ part can be ignored as long as q is finite.

The roots of a quadratic equation $az^2 + bz + c = 0$ are $z^* = [-b \pm \sqrt{b^2 - 4ac}]/(2a)$. You may also be able to find roots by factoring. For a quadratic, for example, look for values of d, e, f, g such that $az^2 + bz + c = (d - ez)(f - gz) = 0$. Then the characteristic roots are $z^* = d/e$ and $z^* = f/g$.